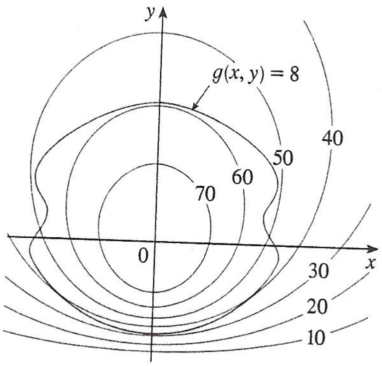
14.8 Exercises

1. Pictured are a contour map of f and a curve with equation $g(x, y) = 8$. Estimate the maximum and minimum values of f subject to the constraint that $g(x, y) = 8$. Explain your reasoning.



2. (a) Use a graphing calculator or computer to graph the circle $x^2 + y^2 = 1$. On the same screen, graph several curves of the form $x^2 + y = c$ until you find two that just touch the circle. What is the significance of the values of c for these two curves?
   (b) Use Lagrange multipliers to find the extreme values of $f(x, y) = x^2 + y$ subject to the constraint $x^2 + y^2 = 1$. Compare your answers with those in part (a).

3–14 Use Lagrange multipliers to find the maximum and minimum values of the function subject to the given constraint.

3. $f(x, y) = x^2 + y^2$; $xy = 1$
4. $f(x, y) = 3x + y$; $x^2 + y^2 = 10$
5. $f(x, y) = y^2 - x^2$; $\frac{1}{4}x^2 + y^2 = 1$
6. $f(x, y) = e^{xy}$; $x^3 + y^3 = 16$
7. $f(x, y, z) = 2x + 2y + z$; $x^2 + y^2 + z^2 = 9$
8. $f(x, y, z) = x^2 + y^2 + z^2$; $x + y + z = 12$
9. $f(x, y, z) = xyz$; $x^2 + 2y^2 + 3z^2 = 6$
10. $f(x, y, z) = x^2y^2z^2$; $x^2 + y^2 + z^2 = 1$
11. $f(x, y, z) = x^2 + y^2 + z^2$; $x^4 + y^4 + z^4 = 1$
12. $f(x, y, z) = x^4 + y^4 + z^4$; $x^2 + y^2 + z^2 = 1$
13. $f(x, y, z, t) = x + y + z + t$; $x^2 + y^2 + z^2 + t^2 = 1$
14. $f(x_1, x_2, \ldots, x_n) = x_1 + x_2 + \cdots + x_n$; $x_1^2 + x_2^2 + \cdots + x_n^2 = 1$

15–18 Find the extreme values of f subject to both constraints.

15. $f(x, y, z) = x + 2y$; $x + y + z = 1$, $y^2 + z^2 = 4$
16. $f(x, y, z) = 3x - y - 3z$; $x + y - z = 0$, $x^2 + 2z^2 = 1$
17. $f(x, y, z) = yz + xy$; $xy = 1$, $y^2 + z^2 = 1$
18. $f(x, y, z) = x^2 + y^2 + z^2$; $x - y = 1$, $y^2 - z^2 = 1$

19–21 Find the extreme values of f on the region described b the inequality.

19. $f(x, y) = x^2 + y^2 + 4x - 4y$, $x^2 + y^2 \leqslant 9$
20. $f(x, y) = 2x^2 + 3y^2 - 4x - 5$, $x^2 + y^2 \leqslant 16$
21. $f(x, y) = e^{-xy}$, $x^2 + 4y^2 \leqslant 1$

22. Consider the problem of maximizing the function $f(x, y) = 2x + 3y$ subject to the constraint $\sqrt{x} + \sqrt{y} = 5$.
   (a) Try using Lagrange multipliers to solve the problem.
   (b) Does $f(25, 0)$ give a larger value than the one in part (a
   (c) Solve the problem by graphing the constraint equation and several level curves of f.
   (d) Explain why the method of Lagrange multipliers fails tc solve the problem.
   (e) What is the significance of $f(9, 4)$?

23. Consider the problem of minimizing the function $f(x, y) =$. on the curve $y^2 + x^4 - x^3 = 0$ (a piriform).
   (a) Try using Lagrange multipliers to solve the problem.
   (b) Show that the minimum value is $f(0, 0) = 0$ but the Lagrange condition $\nabla f(0, 0) = \lambda \nabla g(0, 0)$ is not satisfied for any value of λ.
   (c) Explain why Lagrange multipliers fail to find the minimum value in this case.

CAS 24. (a) If your computer algebra system plots implicitly defined curves, use it to estimate the minimum and maximum values of $f(x, y) = x^3 + y^3 + 3xy$ subject to the constraint $(x - 3)^2 + (y - 3)^2 = 9$ by graphical methods.
   (b) Solve the problem in part (a) with the aid of Lagrange multipliers. Use your CAS to solve the equations numerically. Compare your answers with those in part (a).

25. The total production P of a certain product depends on the amount L of labor used and the amount K of capital investment. In Sections 14.1 and 14.3 we discussed how the Cobb-Douglas model $P = bL^{\alpha}K^{1-\alpha}$ follows from certain economic assumptions, where b and α are positive constants and $\alpha < 1$. If the cost of a unit of labor is m and the cost of a unit of capital is n, and the company can spend only p dollars as its total budget, then maximizing the production P is subject to the constraint $mL + nK = p$. Show that the maximum production occurs when

$$L = \frac{\alpha p}{m} \quad \text{and} \quad K = \frac{(1 - \alpha)p}{n}$$

26. Referring to Exercise 25, we now suppose that the production is fixed at $bL^{\alpha}K^{1-\alpha} = Q$, where Q is a constant. What values of L and K minimize the cost function $C(L, K) = mL + nK$?

27. Use Lagrange multipliers to prove that the rectangle with maximum area that has a given perimeter p is a square.

28. Use Lagrange multipliers to prove that the triangle with maximum area that has a given perimeter p is equilateral.
Hint: Use Heron's formula for the area:

$$A = \sqrt{s(s - x)(s - y)(s - z)}$$

where $s = p/2$ and x, y, z are the lengths of the sides.

29–41 Use Lagrange multipliers to give an alternate solution to the indicated exercise in Section 14.7.

29. Exercise 39 **30.** Exercise 40

31. Exercise 41 **32.** Exercise 42

33. Exercise 43 **34.** Exercise 44

35. Exercise 45 **36.** Exercise 46

37. Exercise 47 **38.** Exercise 48

39. Exercise 49 **40.** Exercise 50

41. Exercise 53

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42. Find the maximum and minimum volumes of a rectangular box whose surface area is 1500 cm^2 and whose total edge length is 200 cm.

43. The plane $x + y + 2z = 2$ intersects the paraboloid $z = x^2 + y^2$ in an ellipse. Find the points on this ellipse that are nearest to and farthest from the origin.

44. The plane $4x - 3y + 8z = 5$ intersects the cone $z^2 = x^2 + y^2$ in an ellipse.
(a) Graph the cone, the plane, and the ellipse.
(b) Use Lagrange multipliers to find the highest and lowest points on the ellipse.

CAS **45–46** Find the maximum and minimum values of f subject to the given constraints. Use a computer algebra system to solve the system of equations that arises in using Lagrange multipliers. (If your CAS finds only one solution, you may need to use additional commands.)

45. $f(x, y, z) = ye^{x-z}$; $9x^2 + 4y^2 + 36z^2 = 36$, $xy + yz = 1$

46. $f(x, y, z) = x + y + z$; $x^2 - y^2 = z$, $x^2 + z^2 = 4$

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47. (a) Find the maximum value of

$$f(x_1, x_2, \ldots, x_n) = \sqrt[n]{x_1 x_2 \cdots x_n}$$

given that $x_1, x_2, \ldots, x_n$ are positive numbers and $x_1 + x_2 + \cdots + x_n = c$, where c is a constant.
(b) Deduce from part (a) that if $x_1, x_2, \ldots, x_n$ are positive numbers, then

$$\sqrt[n]{x_1 x_2 \cdots x_n} \leqslant \frac{x_1 + x_2 + \cdots + x_n}{n}$$

This inequality says that the geometric mean of n numbers is no larger than the arithmetic mean of the numbers. Under what circumstances are these two means equal?

48. (a) Maximize $\Sigma_{i=1}^{n} x_i y_i$ subject to the constraints $\Sigma_{i=1}^{n} x_i^2 = 1$ and $\Sigma_{i=1}^{n} y_i^2 = 1$.
(b) Put

$$x_i = \frac{a_i}{\sqrt{\Sigma a_j^2}} \quad \text{and} \quad y_i = \frac{b_i}{\sqrt{\Sigma b_j^2}}$$

to show that

$$\sum a_i b_i \leqslant \sqrt{\Sigma a_j^2}\,\sqrt{\Sigma b_j^2}$$

for any numbers $a_1, \ldots, a_n, b_1, \ldots, b_n$. This inequality is known as the Cauchy-Schwarz Inequality.

APPLIED PROJECT ROCKET SCIENCE

Many rockets, such as the *Pegasus XL* currently used to launch satellites and the *Saturn V* that first put men on the moon, are designed to use three stages in their ascent into space. A large first stage initially propels the rocket until its fuel is consumed, at which point the stage is jettisoned to reduce the mass of the rocket. The smaller second and third stages function similarly in order to place the rocket's payload into orbit about the earth. (With this design, at least two stages are required in order to reach the necessary velocities, and using three stages has proven to be a good compromise between cost and performance.) Our goal here is to determine the individual masses of the three stages, which are to be designed in such a way as to minimize the total mass of the rocket while enabling it to reach a desired velocity.