$$\boxed{7} \qquad \frac{\partial z}{\partial x} = -\frac{\dfrac{\partial F}{\partial x}}{\dfrac{\partial F}{\partial z}} \qquad \frac{\partial z}{\partial y} = -\frac{\dfrac{\partial F}{\partial y}}{\dfrac{\partial F}{\partial z}}$$

Again, a version of the **Implicit Function Theorem** stipulates conditions under which our assumption is valid: If F is defined within a sphere containing (a, b, c), where $F(a, b, c) = 0$, $F_z(a, b, c) \neq 0$, and F_x, F_y, and F_z are continuous inside the sphere, then the equation $F(x, y, z) = 0$ defines z as a function of x and y near the point (a, b, c) and this function is differentiable, with partial derivatives given by $\boxed{7}$.

EXAMPLE 9 Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ if $x^3 + y^3 + z^3 + 6xyz = 1$.

SOLUTION Let $F(x, y, z) = x^3 + y^3 + z^3 + 6xyz - 1$. Then, from Equations 7, we have

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{3x^2 + 6yz}{3z^2 + 6xy} = -\frac{x^2 + 2yz}{z^2 + 2xy}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{3y^2 + 6xz}{3z^2 + 6xy} = -\frac{y^2 + 2xz}{z^2 + 2xy}$$

The solution to Example 9 should be compared to the one in Example 4 in Section 14.3.

14.5 Exercises

1–6 Use the Chain Rule to find dz/dt or dw/dt.

1. $z = x^2 + y^2 + xy$, $x = \sin t$, $y = e^t$

2. $z = \cos(x + 4y)$, $x = 5t^4$, $y = 1/t$

3. $z = \sqrt{1 + x^2 + y^2}$, $x = \ln t$, $y = \cos t$

4. $z = \tan^{-1}(y/x)$, $x = e^t$, $y = 1 - e^{-t}$

5. $w = xe^{y/z}$, $x = t^2$, $y = 1 - t$, $z = 1 + 2t$

6. $w = \ln\sqrt{x^2 + y^2 + z^2}$, $x = \sin t$, $y = \cos t$, $z = \tan t$

7–12 Use the Chain Rule to find $\partial z/\partial s$ and $\partial z/\partial t$.

7. $z = x^2y^3$, $x = s\cos t$, $y = s\sin t$

8. $z = \arcsin(x - y)$, $x = s^2 + t^2$, $y = 1 - 2st$

9. $z = \sin\theta\cos\phi$, $\theta = st^2$, $\phi = s^2t$

10. $z = e^{x+2y}$, $x = s/t$, $y = t/s$

11. $z = e^r\cos\theta$, $r = st$, $\theta = \sqrt{s^2 + t^2}$

12. $z = \tan(u/v)$, $u = 2s + 3t$, $v = 3s - 2t$

13. If $z = f(x, y)$, where f is differentiable, and

$$\begin{array}{ll} x = g(t) & y = h(t) \\ g(3) = 2 & h(3) = 7 \\ g'(3) = 5 & h'(3) = -4 \\ f_x(2, 7) = 6 & f_y(2, 7) = -8 \end{array}$$

find dz/dt when $t = 3$.

14. Let $W(s, t) = F(u(s, t), v(s, t))$, where F, u, and v are differentiable, and

$$\begin{array}{ll} u(1, 0) = 2 & v(1, 0) = 3 \\ u_s(1, 0) = -2 & v_s(1, 0) = 5 \\ u_t(1, 0) = 6 & v_t(1, 0) = 4 \\ F_u(2, 3) = -1 & F_v(2, 3) = 10 \end{array}$$

Find $W_s(1, 0)$ and $W_t(1, 0)$.

15. Suppose f is a differentiable function of x and y, and $g(u, v) = f(e^u + \sin v, e^u + \cos v)$. Use the table of values to calculate $g_u(0, 0)$ and $g_v(0, 0)$.

	f	g	f_x	f_y
$(0, 0)$	3	6	4	8
$(1, 2)$	6	3	2	5

16. Suppose f is a differentiable function of x and y, and $g(r, s) = f(2r - s, s^2 - 4r)$. Use the table of values in Exercise 15 to calculate $g_r(1, 2)$ and $g_s(1, 2)$.

17–20 Use a tree diagram to write out the Chain Rule for the given case. Assume all functions are differentiable.

17. $u = f(x, y)$, where $x = x(r, s, t)$, $y = y(r, s, t)$

18. $R = f(x, y, z, t)$, where $x = x(u, v, w)$, $y = y(u, v, w)$, $z = z(u, v, w)$, $t = t(u, v, w)$

19. $w = f(r, s, t)$, where $r = r(x, y)$, $s = s(x, y)$, $t = t(x, y)$

20. $t = f(u, v, w)$, where $u = u(p, q, r, s)$, $v = v(p, q, r, s)$, $w = w(p, q, r, s)$

21–26 Use the Chain Rule to find the indicated partial derivatives.

21. $z = x^4 + x^2y$, $x = s + 2t - u$, $y = stu^2$;
$\frac{\partial z}{\partial s}, \frac{\partial z}{\partial t}, \frac{\partial z}{\partial u}$ when $s = 4$, $t = 2$, $u = 1$

22. $T = \frac{v}{2u + v}$, $u = pq\sqrt{r}$, $v = p\sqrt{q}r$;
$\frac{\partial T}{\partial p}, \frac{\partial T}{\partial q}, \frac{\partial T}{\partial r}$ when $p = 2$, $q = 1$, $r = 4$

23. $w = xy + yz + zx$, $x = r\cos\theta$, $y = r\sin\theta$, $z = r\theta$;
$\frac{\partial w}{\partial r}, \frac{\partial w}{\partial \theta}$ when $r = 2$, $\theta = \pi/2$

24. $P = \sqrt{u^2 + v^2 + w^2}$, $u = xe^y$, $v = ye^x$, $w = e^{xy}$;
$\frac{\partial P}{\partial x}, \frac{\partial P}{\partial y}$ when $x = 0$, $y = 2$

25. $N = \frac{p + q}{p + r}$, $p = u + vw$, $q = v + uw$, $r = w + uv$;
$\frac{\partial N}{\partial u}, \frac{\partial N}{\partial v}, \frac{\partial N}{\partial w}$ when $u = 2$, $v = 3$, $w = 4$

26. $u = xe^{ty}$, $x = \alpha^2\beta$, $y = \beta^2\gamma$, $t = \gamma^2\alpha$;
$\frac{\partial u}{\partial \alpha}, \frac{\partial u}{\partial \beta}, \frac{\partial u}{\partial \gamma}$ when $\alpha = -1$, $\beta = 2$, $\gamma = 1$

27–30 Use Equation 6 to find dy/dx.

27. $y\cos x = x^2 + y^2$

28. $\cos(xy) = 1 + \sin y$

29. $\tan^{-1}(x^2y) = x + xy^2$

30. $e^y \sin x = x + xy$

31–34 Use Equations 7 to find $\partial z/\partial x$ and $\partial z/\partial y$.

31. $x^2 + 2y^2 + 3z^2 = 1$

32. $x^2 - y^2 + z^2 - 2z = 4$

33. $e^z = xyz$

34. $yz + x\ln y = z^2$

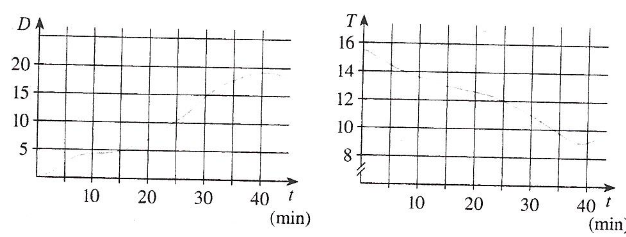
35. The temperature at a point (x, y) is $T(x, y)$, measured in degrees Celsius. A bug crawls so that its position after t seconds is given by $x = \sqrt{1 + t}$, $y = 2 + \frac{1}{3}t$, where x and y are measured in centimeters. The temperature function satisfies $T_x(2, 3) = 4$ and $T_y(2, 3) = 3$. How fast is the temperature rising on the bug's path after 3 seconds?

36. Wheat production W in a given year depends on the average temperature T and the annual rainfall R. Scientists estimate that the average temperature is rising at a rate of 0.15°C/year and rainfall is decreasing at a rate of 0.1 cm/year. They also estimate that, at current production levels, $\partial W/\partial T = -2$ and $\partial W/\partial R = 8$.
(a) What is the significance of the signs of these partial derivatives?
(b) Estimate the current rate of change of wheat production, dW/dt.

37. The speed of sound traveling through ocean water with salinity 35 parts per thousand has been modeled by the equation

$$C = 1449.2 + 4.6T - 0.055T^2 + 0.00029T^3 + 0.016D$$

where C is the speed of sound (in meters per second), T is the temperature (in degrees Celsius), and D is the depth below the ocean surface (in meters). A scuba diver began a leisurely dive into the ocean water; the diver's depth and the surrounding water temperature over time are recorded in the following graphs. Estimate the rate of change (with respect to time) of the speed of sound through the ocean water experienced by the diver 20 minutes into the dive. What are the units?



38. The radius of a right circular cone is increasing at a rate of 1.8 in/s while its height is decreasing at a rate of 2.5 in/s. At what rate is the volume of the cone changing when the radius is 120 in. and the height is 140 in.?

39. The length ℓ, width w, and height h of a box change with time. At a certain instant the dimensions are $\ell = 1$ m and $w = h = 2$ m, and ℓ and w are increasing at a rate of 2 m/s while h is decreasing at a rate of 3 m/s. At that instant find the rates at which the following quantities are changing.
(a) The volume
(b) The surface area
(c) The length of a diagonal

40. The voltage V in a simple electrical circuit is slowly decreasing as the battery wears out. The resistance R is slowly increasing as the resistor heats up. Use Ohm's Law, $V = IR$, to find how the current I is changing at the moment when $R = 400\ \Omega$, $I = 0.08$ A, $dV/dt = -0.01$ V/s, and $dR/dt = 0.03\ \Omega$/s.

41. The pressure of 1 mole of an ideal gas is increasing at a rate of 0.05 kPa/s and the temperature is increasing at a rate of 0.15 K/s. Use the equation in Example 2 to find the rate of change of the volume when the pressure is 20 kPa and the temperature is 320 K.

42. A manufacturer has modeled its yearly production function P (the value of its entire production in millions of dollars) as a Cobb-Douglas function

$$P(L, K) = 1.47L^{0.65}K^{0.35}$$

where L is the number of labor hours (in thousands) and K is