Find the radius of convergence and interval of convergence of the series

$$\sum_{n=0}^{\infty} \frac{n(x+2)^n}{3^{n+1}}$$

If $a_n = n(x+2)^n/3^{n+1}$, then

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)(x+2)^{n+1}}{3^{n+2}} \cdot \frac{3^{n+1}}{n(x+2)^n}\right|$$

$$= \left(1 + \frac{1}{n}\right)\frac{|x+2|}{3} \to \frac{|x+2|}{3} \quad \text{as } n \to \infty$$

Using the Ratio Test, we see that the series converges if $|x+2|/3 < 1$ and it diverges if $|x+2|/3 > 1$. So it converges if $|x+2| < 3$ and diverges if $|x+2| > 3$. Thus the radius of convergence is $R = 3$.

The inequality $|x+2| < 3$ can be written as $-5 < x < 1$, so we test the series at the endpoints -5 and 1. When $x = -5$, the series is

$$\sum_{n=0}^{\infty} \frac{n(-3)^n}{3^{n+1}} = \tfrac{1}{3}\sum_{n=0}^{\infty} (-1)^n n$$

which diverges by the Test for Divergence [$(-1)^n n$ doesn't converge to 0]. When $x = 1$, the series is

$$\sum_{n=0}^{\infty} \frac{n(3)^n}{3^{n+1}} = \tfrac{1}{3}\sum_{n=0}^{\infty} n$$

which also diverges by the Test for Divergence. Thus the series converges only when $-5 < x < 1$, so the interval of convergence is $(-5, 1)$.

11.8 Exercises

1. What is a power series?

2. (a) What is the radius of convergence of a power series? How do you find it?
(b) What is the interval of convergence of a power series? How do you find it?

3–28 Find the radius of convergence and interval of convergence of the series.

3. $\sum_{n=1}^{\infty} (-1)^n n x^n$

4. $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{\sqrt[3]{n}}$

5. $\sum_{n=1}^{\infty} \frac{x^n}{2n-1}$

6. $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n^2}$

7. $\sum_{n=0}^{\infty} \frac{x^n}{n!}$

8. $\sum_{n=1}^{\infty} n^n x^n$

9. $\sum_{n=1}^{\infty} (-1)^n \frac{n^2 x^n}{2^n}$

10. $\sum_{n=1}^{\infty} \frac{10^n x^n}{n^3}$

11. $\sum_{n=1}^{\infty} \frac{(-3)^n}{n\sqrt{n}} x^n$

12. $\sum_{n=1}^{\infty} \frac{x^n}{n3^n}$

13. $\sum_{n=2}^{\infty} (-1)^n \frac{x^n}{4^n \ln n}$

14. $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$

15. $\sum_{n=0}^{\infty} \frac{(x-2)^n}{n^2+1}$

16. $\sum_{n=0}^{\infty} (-1)^n \frac{(x-3)^n}{2n+1}$

17. $\sum_{n=1}^{\infty} \frac{3^n(x+4)^n}{\sqrt{n}}$

18. $\sum_{n=1}^{\infty} \frac{n}{4^n}(x+1)^n$

19. $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n^n}$

20. $\sum_{n=1}^{\infty} \frac{(2x-1)^n}{5^n\sqrt{n}}$

21. $\sum_{n=1}^{\infty} \frac{n}{b^n}(x-a)^n, \quad b>0$

22. $\sum_{n=2}^{\infty} \frac{b^n}{\ln n}(x-a)^n, \quad b>0$

23. $\sum_{n=1}^{\infty} n!(2x-1)^n$

24. $\sum_{n=1}^{\infty} \frac{n^2x^n}{2 \cdot 4 \cdot 6 \cdot \cdots \cdot (2n)}$

25. $\sum_{n=1}^{\infty} \frac{(5x-4)^n}{n^3}$

26. $\sum_{n=2}^{\infty} \frac{x^{2n}}{n(\ln n)^2}$

27. $\sum_{n=1}^{\infty} \frac{x^n}{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}$

28. $\sum_{n=1}^{\infty} \frac{n!x^n}{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}$

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29. If $\Sigma_{n=0}^{\infty} c_n 4^n$ is convergent, does it follow that the following series are convergent?

(a) $\sum_{n=0}^{\infty} c_n(-2)^n$ (b) $\sum_{n=0}^{\infty} c_n(-4)^n$

30. Suppose that $\Sigma_{n=0}^{\infty} c_n x^n$ converges when $x=-4$ and diverges when $x=6$. What can be said about the convergence or divergence of the following series?

(a) $\sum_{n=0}^{\infty} c_n$ (b) $\sum_{n=0}^{\infty} c_n 8^n$

(c) $\sum_{n=0}^{\infty} c_n(-3)^n$ (d) $\sum_{n=0}^{\infty} (-1)^n c_n 9^n$

31. If k is a positive integer, find the radius of convergence of the series

$$\sum_{n=0}^{\infty} \frac{(n!)^k}{(kn)!} x^n$$

32. Let p and q be real numbers with $p<q$. Find a power series whose interval of convergence is

(a) (p, q) (b) $(p, q]$

(c) $[p, q)$ (d) $[p, q]$

33. Is it possible to find a power series whose interval of convergence is $[0, \infty)$? Explain.

34. Graph the first several partial sums $s_n(x)$ of the series $\Sigma_{n=0}^{\infty} x^n$, together with the sum function $f(x)=1/(1-x)$, on a common screen. On what interval do these partial sums appear to be converging to $f(x)$?

35. The function J_1 defined by

$$J_1(x)=\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!(n+1)!2^{2n+1}}$$

is called the *Bessel function of order 1.*

(a) Find its domain.

(b) Graph the first several partial sums on a common screen.

CAS (c) If your CAS has built-in Bessel functions, graph J_1 on the same screen as the partial sums in part (b) and observe how the partial sums approximate J_1.

36. The function A defined by

$$A(x)=1+\frac{x^3}{2 \cdot 3}+\frac{x^6}{2 \cdot 3 \cdot 5 \cdot 6}+\frac{x^9}{2 \cdot 3 \cdot 5 \cdot 6 \cdot 8 \cdot 9}+\cdots$$

is called an *Airy function* after the English mathematician and astronomer Sir George Airy (1801–1892).

(a) Find the domain of the Airy function.

(b) Graph the first several partial sums on a common screen.

CAS (c) If your CAS has built-in Airy functions, graph A on the same screen as the partial sums in part (b) and observe how the partial sums approximate A.

37. A function f is defined by

$$f(x)=1+2x+x^2+2x^3+x^4+\cdots$$

that is, its coefficients are $c_{2n}=1$ and $c_{2n+1}=2$ for all $n \geqslant 0$. Find the interval of convergence of the series and find an explicit formula for $f(x)$.

38. If $f(x)=\Sigma_{n=0}^{\infty} c_n x^n$, where $c_{n+4}=c_n$ for all $n \geqslant 0$, find the interval of convergence of the series and a formula for $f(x)$.

39. Show that if $\lim_{n\to\infty} \sqrt[n]{|c_n|}=c$, where $c \neq 0$, then the radius of convergence of the power series $\Sigma c_n x^n$ is $R=1/c$.

40. Suppose that the power series $\Sigma c_n(x-a)^n$ satisfies $c_n \neq 0$ for all n. Show that if $\lim_{n\to\infty} |c_n/c_{n+1}|$ exists, then it is equal to the radius of convergence of the power series.

41. Suppose the series $\Sigma c_n x^n$ has radius of convergence 2 and the series $\Sigma d_n x^n$ has radius of convergence 3. What is the radius of convergence of the series $\Sigma (c_n+d_n)x^n$?

42. Suppose that the radius of convergence of the power series $\Sigma c_n x^n$ is R. What is the radius of convergence of the power series $\Sigma c_n x^{2n}$?

11.9 Representations of Functions as Power Series

In this section we learn how to represent certain types of functions as sums of power series by manipulating geometric series or by differentiating or integrating such a series. You might wonder why we would ever want to express a known function as a sum of infinitely many terms. We will see later that this strategy is useful for integrating functions that don't have elementary antiderivatives, for solving differential equations, and for approximating func-