A third method for visualizing the twisted cubic is to realize that it also lies on the cylinder $z = x^3$. So it can be viewed as the curve of intersection of the cylinders $y = x^2$ and $z = x^3$. (See Figure 11.)

TEC Visual 13.1C shows how curves arise as intersections of surfaces.

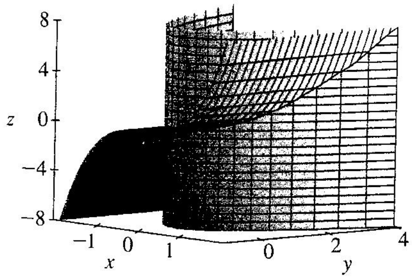
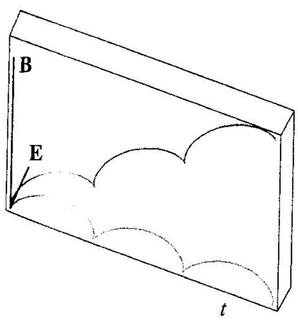


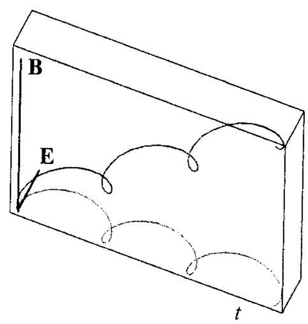
FIGURE 11

Some computer algebra systems provide us with a clearer picture of a space curve by enclosing it in a tube. Such a plot enables us to see whether one part of a curve passes in front of or behind another part of the curve. For example, Figure 13 shows the curve of Figure 12(b) as rendered by the `tubeplot` command in Maple.

We have seen that an interesting space curve, the helix, occurs in the model of DNA. Another notable example of a space curve in science is the trajectory of a positively charged particle in orthogonally oriented electric and magnetic fields **E** and **B**. Depending on the initial velocity given the particle at the origin, the path of the particle is either a space curve whose projection on the horizontal plane is the cycloid we studied in Section 10.1 [Figure 12(a)] or a curve whose projection is the trochoid investigated in Exercise 40 in Section 10.1 [Figure 12(b)].



(a) $\mathbf{r}(t) = \langle t - \sin t, 1 - \cos t, t \rangle$



(b) $\mathbf{r}(t) = \left\langle t - \frac{3}{2}\sin t, 1 - \frac{3}{2}\cos t, t \right\rangle$

FIGURE 12
Motion of a charged particle in orthogonally oriented electric and magnetic fields

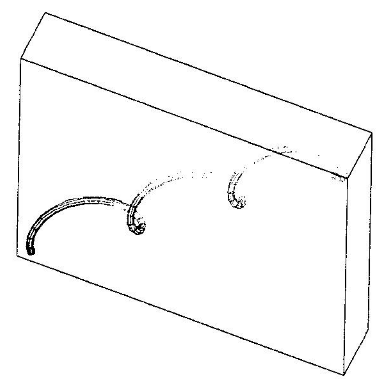


FIGURE 13

For further details concerning the physics involved and animations of the trajectories of the particles, see the following web sites:

- www.phy.ntnu.edu.tw/java/emField/emField.html
- www.physics.ucla.edu/plasma-exp/Beam/

13.1 Exercises

1–2 Find the domain of the vector function.

1. $\mathbf{r}(t) = \left\langle \sqrt{4 - t^2}, e^{-3t}, \ln(t + 1) \right\rangle$

2. $\mathbf{r}(t) = \dfrac{t - 2}{t + 2}\mathbf{i} + \sin t\,\mathbf{j} + \ln(9 - t^2)\,\mathbf{k}$

3–6 Find the limit.

3. $\displaystyle\lim_{t \to 0} \left(e^{-3t}\mathbf{i} + \frac{t^2}{\sin^2 t}\mathbf{j} + \cos 2t\,\mathbf{k} \right)$

4. $\displaystyle\lim_{t \to 1} \left(\frac{t^2 - t}{t - 1}\mathbf{i} + \sqrt{t + 8}\,\mathbf{j} + \frac{\sin \pi t}{\ln t}\mathbf{k} \right)$

5. $\lim_{t \to \infty} \left\langle \frac{1 + t^2}{1 - t^2}, \tan^{-1} t, \frac{1 - e^{-2t}}{t} \right\rangle$

6. $\lim_{t \to \infty} \left\langle te^{-t}, \frac{t^3 + t}{2t^3 - 1}, t \sin \frac{1}{t} \right\rangle$

7–14 Sketch the curve with the given vector equation. Indicate with an arrow the direction in which t increases.

7. $\mathbf{r}(t) = \langle \sin t, t \rangle$

8. $\mathbf{r}(t) = \langle t^3, t^2 \rangle$

9. $\mathbf{r}(t) = \langle t, 2 - t, 2t \rangle$

10. $\mathbf{r}(t) = \langle \sin \pi t, t, \cos \pi t \rangle$

11. $\mathbf{r}(t) = \langle 1, \cos t, 2 \sin t \rangle$

12. $\mathbf{r}(t) = t^2 \mathbf{i} + t \mathbf{j} + 2 \mathbf{k}$

13. $\mathbf{r}(t) = t^2 \mathbf{i} + t^4 \mathbf{j} + t^6 \mathbf{k}$

14. $\mathbf{r}(t) = \cos t \, \mathbf{i} - \cos t \, \mathbf{j} + \sin t \, \mathbf{k}$

15–16 Draw the projections of the curve on the three coordinate planes. Use these projections to help sketch the curve.

15. $\mathbf{r}(t) = \langle t, \sin t, 2 \cos t \rangle$

16. $\mathbf{r}(t) = \langle t, t, t^2 \rangle$

17–20 Find a vector equation and parametric equations for the line segment that joins P to Q.

17. $P(2, 0, 0)$, $Q(6, 2, -2)$

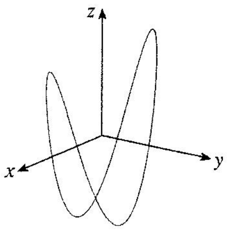
18. $P(-1, 2, -2)$, $Q(-3, 5, 1)$

19. $P(0, -1, 1)$, $Q(\frac{1}{2}, \frac{1}{3}, \frac{1}{4})$

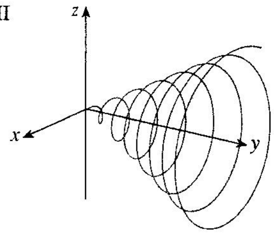
20. $P(a, b, c)$, $Q(u, v, w)$

21–26 Match the parametric equations with the graphs (labeled I–VI). Give reasons for your choices.

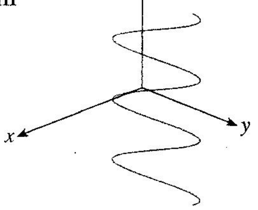
I



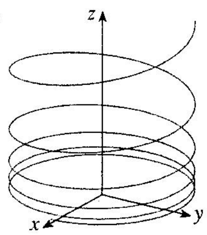
II



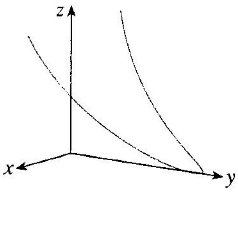
III



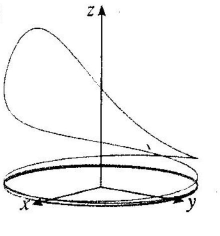
IV



V



VI



21. $x = t \cos t$, $y = t$, $z = t \sin t$, $t \geq 0$

22. $x = \cos t$, $y = \sin t$, $z = 1/(1 + t^2)$

23. $x = t$, $y = 1/(1 + t^2)$, $z = t^2$

24. $x = \cos t$, $y = \sin t$, $z = \cos 2t$

25. $x = \cos 8t$, $y = \sin 8t$, $z = e^{0.8t}$, $t \geq 0$

26. $x = \cos^2 t$, $y = \sin^2 t$, $z = t$

27. Show that the curve with parametric equations $x = t \cos t$, $y = t \sin t$, $z = t$ lies on the cone $z^2 = x^2 + y^2$, and use this fact to help sketch the curve.

28. Show that the curve with parametric equations $x = \sin t$, $y = \cos t$, $z = \sin^2 t$ is the curve of intersection of the surfaces $z = x^2$ and $x^2 + y^2 = 1$. Use this fact to help sketch the curve.

29. At what points does the curve $\mathbf{r}(t) = t \, \mathbf{i} + (2t - t^2) \, \mathbf{k}$ intersect the paraboloid $z = x^2 + y^2$?

30. At what points does the helix $\mathbf{r}(t) = \langle \sin t, \cos t, t \rangle$ intersect the sphere $x^2 + y^2 + z^2 = 5$?

31–35 Use a computer to graph the curve with the given vector equation. Make sure you choose a parameter domain and viewpoints that reveal the true nature of the curve.

31. $\mathbf{r}(t) = \langle \cos t \sin 2t, \sin t \sin 2t, \cos 2t \rangle$

32. $\mathbf{r}(t) = \langle t^2, \ln t, t \rangle$

33. $\mathbf{r}(t) = \langle t, t \sin t, t \cos t \rangle$

34. $\mathbf{r}(t) = \langle t, e^t, \cos t \rangle$

35. $\mathbf{r}(t) = \langle \cos 2t, \cos 3t, \cos 4t \rangle$

36. Graph the curve with parametric equations $x = \sin t$, $y = \sin 2t$, $z = \cos 4t$. Explain its shape by graphing its projections onto the three coordinate planes.

37. Graph the curve with parametric equations

$$x = (1 + \cos 16t) \cos t$$
$$y = (1 + \cos 16t) \sin t$$
$$z = 1 + \cos 16t$$

Explain the appearance of the graph by showing that it lies on a cone.

38. Graph the curve with parametric equations

$$x = \sqrt{1 - 0.25 \cos^2 10t} \, \cos t$$
$$y = \sqrt{1 - 0.25 \cos^2 10t} \, \sin t$$
$$z = 0.5 \cos 10t$$

Explain the appearance of the graph by showing that it lies on a sphere.