This example demonstrates one way in which power series representations are useful. Integrating $1/(1 + x^7)$ by hand is incredibly difficult. Different computer algebra systems return different forms of the answer, but they are all extremely complicated. (If you have a CAS, try it yourself.) The infinite series answer that we obtain in Example 8(a) is actually much easier to deal with than the finite answer provided by a CAS.

Now we integrate term by term:

$$\int \frac{1}{1+x^7}\,dx = \int \sum_{n=0}^{\infty} (-1)^n x^{7n}\,dx = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{7n+1}}{7n+1}$$

$$= C + x - \frac{x^8}{8} + \frac{x^{15}}{15} - \frac{x^{22}}{22} + \cdots$$

This series converges for $|-x^7| < 1$, that is, for $|x| < 1$.

(b) In applying the Fundamental Theorem of Calculus, it doesn't matter which antiderivative we use, so let's use the antiderivative from part (a) with $C = 0$:

$$\int_0^{0.5} \frac{1}{1+x^7}\,dx = \left[ x - \frac{x^8}{8} + \frac{x^{15}}{15} - \frac{x^{22}}{22} + \cdots \right]_0^{1/2}$$

$$= \frac{1}{2} - \frac{1}{8 \cdot 2^8} + \frac{1}{15 \cdot 2^{15}} - \frac{1}{22 \cdot 2^{22}} + \cdots + \frac{(-1)^n}{(7n+1)2^{7n+1}} + \cdots$$

This infinite series is the exact value of the definite integral, but since it is an alternating series, we can approximate the sum using the Alternating Series Estimation Theorem. If we stop adding after the term with $n = 3$, the error is smaller than the term with $n = 4$:

$$\frac{1}{29 \cdot 2^{29}} \approx 6.4 \times 10^{-11}$$

So we have

$$\int_0^{0.5} \frac{1}{1+x^7}\,dx \approx \frac{1}{2} - \frac{1}{8 \cdot 2^8} + \frac{1}{15 \cdot 2^{15}} - \frac{1}{22 \cdot 2^{22}} \approx 0.49951374$$

## Exercises

**1.** If the radius of convergence of the power series $\sum_{n=0}^{\infty} c_n x^n$ is 10, what is the radius of convergence of the series $\sum_{n=1}^{\infty} n c_n x^{n-1}$? Why?

**2.** Suppose you know that the series $\sum_{n=0}^{\infty} b_n x^n$ converges for $|x| < 2$. What can you say about the following series? Why?

$$\sum_{n=0}^{\infty} \frac{b_n}{n+1} x^{n+1}$$

Find a power series representation for the function and determine the interval of convergence.

**3.** $f(x) = \dfrac{1}{1+x}$

**4.** $f(x) = \dfrac{5}{1-4x^2}$

$f(x) = \dfrac{2}{3-x}$

**6.** $f(x) = \dfrac{1}{x+10}$

**7.** $f(x) = \dfrac{x}{9+x^2}$

$f(x) = \dfrac{x}{2x^2+1}$

**9.** $f(x) = \dfrac{1+x}{1-x}$

**10.** $f(x) = \dfrac{x^2}{a^3-x^3}$

Express the function as the sum of a power series by first using partial fractions. Find the interval of convergence.

**11.** $f(x) = \dfrac{3}{x^2-x-2}$

**12.** $f(x) = \dfrac{x+2}{2x^2-x-1}$

(a) Use differentiation to find a power series representation for

$$f(x) = \frac{1}{(1+x)^2}$$

What is the radius of convergence?

(b) Use part (a) to find a power series for

$$f(x) = \frac{1}{(1+x)^3}$$

(c) Use part (b) to find a power series for

$$f(x) = \frac{x^2}{(1+x)^3}$$

**14.** (a) Use Equation 1 to find a power series representation for $f(x) = \ln(1 - x)$. What is the radius of convergence?
(b) Use part (a) to find a power series for $f(x) = x\ln(1 - x)$.
(c) By putting $x = \frac{1}{2}$ in your result from part (a), express ln 2 as the sum of an infinite series.

**15–20** Find a power series representation for the function and determine the radius of convergence.

**15.** $f(x) = \ln(5 - x)$

**16.** $f(x) = x^2 \tan^{-1}(x^3)$

**17.** $f(x) = \dfrac{x}{(1+4x)^2}$

**18.** $f(x) = \left(\dfrac{x}{2-x}\right)^3$

**19.** $f(x) = \dfrac{1+x}{(1-x)^2}$

**20.** $f(x) = \dfrac{x^2+x}{(1-x)^3}$

**21–24** Find a power series representation for $f$, and graph $f$ and several partial sums $s_n(x)$ on the same screen. What happens as $n$ increases?

**21.** $f(x) = \dfrac{x}{x^2+16}$

**22.** $f(x) = \ln(x^2 + 4)$

**23.** $f(x) = \ln\left(\dfrac{1+x}{1-x}\right)$

**24.** $f(x) = \tan^{-1}(2x)$

**25–28** Evaluate the indefinite integral as a power series. What is the radius of convergence?

**25.** $\displaystyle\int \frac{t}{1-t^8}\,dt$

**26.** $\displaystyle\int \frac{t}{1+t^3}\,dt$

**27.** $\displaystyle\int x^2\ln(1+x)\,dx$

**28.** $\displaystyle\int \frac{\tan^{-1}x}{x}\,dx$

**29–32** Use a power series to approximate the definite integral to six decimal places.

**29.** $\displaystyle\int_0^{0.2} \frac{1}{1+x^5}\,dx$

**30.** $\displaystyle\int_0^{0.4} \ln(1+x^4)\,dx$

**31.** $\displaystyle\int_0^{0.1} x\arctan(3x)\,dx$

**32.** $\displaystyle\int_0^{0.3} \frac{x^2}{1+x^4}\,dx$

**33.** Use the result of Example 7 to compute arctan 0.2 correct to five decimal places.

**34.** Show that the function

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

is a solution of the differential equation

$$f''(x) + f(x) = 0$$

**35.** (a) Show that $J_0$ (the Bessel function of order 0 given in Example 4) satisfies the differential equation

$$x^2 J_0''(x) + xJ_0'(x) + x^2 J_0(x) = 0$$

(b) Evaluate $\int_0^1 J_0(x)\,dx$ correct to three decimal places.

**36.** The Bessel function of order 1 is defined by

$$J_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!(n+1)!2^{2n+1}}$$

(a) Show that $J_1$ satisfies the differential equation

$$x^2 J_1''(x) + xJ_1'(x) + (x^2 - 1)J_1(x) = 0$$

(b) Show that $J_0'(x) = -J_1(x)$.

**37.** (a) Show that the function

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

is a solution of the differential equation

$$f'(x) = f(x)$$

(b) Show that $f(x) = e^x$.

**38.** Let $f_n(x) = (\sin nx)/n^2$. Show that the series $\Sigma f_n(x)$ converges for all values of $x$ but the series of derivatives $\Sigma f_n'(x)$ diverges when $x = 2n\pi$, $n$ an integer. For what values of $x$ does the series $\Sigma f_n''(x)$ converge?

**39.** Let

$$f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}$$

Find the intervals of convergence for $f$, $f'$, and $f''$.

**40.** (a) Starting with the geometric series $\Sigma_{n=0}^{\infty} x^n$, find the sum of the series

$$\sum_{n=1}^{\infty} nx^{n-1} \qquad |x| < 1$$

(b) Find the sum of each of the following series.

(i) $\displaystyle\sum_{n=1}^{\infty} nx^n,\quad |x| < 1$ (ii) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2^n}$