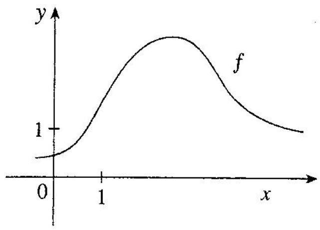
11.10 Exercises

1. If $f(x) = \sum_{n=0}^{\infty} b_n(x-5)^n$ for all x, write a formula for b_8.

2. The graph of f is shown.



(a) Explain why the series

$$1.6 - 0.8(x-1) + 0.4(x-1)^2 - 0.1(x-1)^3 + \cdots$$

is *not* the Taylor series of f centered at 1.

(b) Explain why the series

$$2.8 + 0.5(x-2) + 1.5(x-2)^2 - 0.1(x-2)^3 + \cdots$$

is *not* the Taylor series of f centered at 2.

3. If $f^{(n)}(0) = (n+1)!$ for $n = 0, 1, 2, \ldots$, find the Maclaurin series for f and its radius of convergence.

4. Find the Taylor series for f centered at 4 if

$$f^{(n)}(4) = \frac{(-1)^n n!}{3^n(n+1)}$$

What is the radius of convergence of the Taylor series?

5–12 Find the Maclaurin series for $f(x)$ using the definition of a Maclaurin series. [Assume that f has a power series expansion. Do not show that $R_n(x) \to 0$.] Also find the associated radius of convergence.

5. $f(x) = (1-x)^{-2}$
6. $f(x) = \ln(1+x)$
7. $f(x) = \sin \pi x$
8. $f(x) = e^{-2x}$
9. $f(x) = 2^x$
10. $f(x) = x \cos x$
11. $f(x) = \sinh x$
12. $f(x) = \cosh x$

13–20 Find the Taylor series for $f(x)$ centered at the given value of a. [Assume that f has a power series expansion. Do not show that $R_n(x) \to 0$.] Also find the associated radius of convergence.

13. $f(x) = x^4 - 3x^2 + 1, \quad a = 1$
14. $f(x) = x - x^3, \quad a = -2$
15. $f(x) = \ln x, \quad a = 2$
16. $f(x) = 1/x, \quad a = -3$
17. $f(x) = e^{2x}, \quad a = 3$
18. $f(x) = \sin x, \quad a = \pi/2$
19. $f(x) = \cos x, \quad a = \pi$
20. $f(x) = \sqrt{x}, \quad a = 16$

21. Prove that the series obtained in Exercise 7 represents $\sin \pi x$ for all x.

22. Prove that the series obtained in Exercise 18 represents $\sin x$ for all x.

23. Prove that the series obtained in Exercise 11 represents $\sinh x$ for all x.

24. Prove that the series obtained in Exercise 12 represents $\cosh x$ for all x.

25–28 Use the binomial series to expand the function as a power series. State the radius of convergence.

25. $\sqrt[4]{1-x}$
26. $\sqrt[3]{8+x}$
27. $\dfrac{1}{(2+x)^3}$
28. $(1-x)^{2/3}$

29–38 Use a Maclaurin series in Table 1 to obtain the Maclaurin series for the given function.

29. $f(x) = \sin \pi x$
30. $f(x) = \cos(\pi x/2)$
31. $f(x) = e^x + e^{2x}$
32. $f(x) = e^x + 2e^{-x}$
33. $f(x) = x\cos(\frac{1}{2}x^2)$
34. $f(x) = x^2 \ln(1+x^3)$
35. $f(x) = \dfrac{x}{\sqrt{4+x^2}}$
36. $f(x) = \dfrac{x^2}{\sqrt{2+x}}$
37. $f(x) = \sin^2 x$ $\left[\textit{Hint: } \text{Use } \sin^2 x = \frac{1}{2}(1 - \cos 2x).\right]$
38. $f(x) = \begin{cases} \dfrac{x - \sin x}{x^3} & \text{if } x \neq 0 \\ \frac{1}{6} & \text{if } x = 0 \end{cases}$

39–42 Find the Maclaurin series of f (by any method) and its radius of convergence. Graph f and its first few Taylor polynomials on the same screen. What do you notice about the relationship between these polynomials and f?

39. $f(x) = \cos(x^2)$
40. $f(x) = e^{-x^2} + \cos x$
41. $f(x) = xe^{-x}$
42. $f(x) = \tan^{-1}(x^3)$

43. Use the Maclaurin series for $\cos x$ to compute $\cos 5°$ correct to five decimal places.

44. Use the Maclaurin series for e^x to calculate $1/\sqrt[10]{e}$ correct to five decimal places.

45. (a) Use the binomial series to expand $1/\sqrt{1-x^2}$.
(b) Use part (a) to find the Maclaurin series for $\sin^{-1}x$.

46. (a) Expand $1/\sqrt[4]{1+x}$ as a power series.
(b) Use part (a) to estimate $1/\sqrt[4]{1.1}$ correct to three decimal places.

Graphing calculator or computer required   1. Homework Hints available at stewartcalculus.com

47–50 Evaluate the indefinite integral as an infinite series.

47. $\int x \cos(x^3)\, dx$

48. $\int \frac{e^x - 1}{x}\, dx$

49. $\int \frac{\cos x - 1}{x}\, dx$

50. $\int \arctan(x^2)\, dx$

51–54 Use series to approximate the definite integral to within the indicated accuracy.

51. $\int_0^{1/2} x^3 \arctan x\, dx$ (four decimal places)

52. $\int_0^1 \sin(x^4)\, dx$ (four decimal places)

53. $\int_0^{0.4} \sqrt{1 + x^4}\, dx$ $(|\text{error}| < 5 \times 10^{-6})$

54. $\int_0^{0.5} x^2 e^{-x^2}\, dx$ $(|\text{error}| < 0.001)$

55–57 Use series to evaluate the limit.

55. $\lim_{x \to 0} \frac{x - \ln(1 + x)}{x^2}$

56. $\lim_{x \to 0} \frac{1 - \cos x}{1 + x - e^x}$

57. $\lim_{x \to 0} \frac{\sin x - x + \frac{1}{6}x^3}{x^5}$

58. Use the series in Example 13(b) to evaluate

$$\lim_{x \to 0} \frac{\tan x - x}{x^3}$$

We found this limit in Example 4 in Section 4.4 using l'Hospital's Rule three times. Which method do you prefer?

59–62 Use multiplication or division of power series to find the first three nonzero terms in the Maclaurin series for each function.

59. $y = e^{-x^2} \cos x$

60. $y = \sec x$

61. $y = \frac{x}{\sin x}$

62. $y = e^x \ln(1 + x)$

63–70 Find the sum of the series.

63. $\sum_{n=0}^{\infty} (-1)^n \frac{x^{4n}}{n!}$

64. $\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{6^{2n}(2n)!}$

65. $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{3^n}{n\, 5^n}$

66. $\sum_{n=0}^{\infty} \frac{3^n}{5^n n!}$

67. $\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{4^{2n+1}(2n + 1)!}$

68. $1 - \ln 2 + \frac{(\ln 2)^2}{2!} - \frac{(\ln 2)^3}{3!} + \cdots$

69. $3 + \frac{9}{2!} + \frac{27}{3!} + \frac{81}{4!} + \cdots$

70. $\frac{1}{1 \cdot 2} - \frac{1}{3 \cdot 2^3} + \frac{1}{5 \cdot 2^5} - \frac{1}{7 \cdot 2^7} + \cdots$

71. Show that if p is an nth-degree polynomial, then

$$p(x + 1) = \sum_{i=0}^{n} \frac{p^{(i)}(x)}{i!}$$

72. If $f(x) = (1 + x^3)^{30}$, what is $f^{(58)}(0)$?

73. Prove Taylor's Inequality for $n = 2$, that is, prove that if $|f'''(x)| \leqslant M$ for $|x - a| \leqslant d$, then

$$|R_2(x)| \leqslant \frac{M}{6}|x - a|^3 \qquad \text{for } |x - a| \leqslant d$$

74. (a) Show that the function defined by

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is not equal to its Maclaurin series.

(b) Graph the function in part (a) and comment on its behavior near the origin.

75. Use the following steps to prove 17.

(a) Let $g(x) = \sum_{n=0}^{\infty} \binom{k}{n} x^n$. Differentiate this series to show that

$$g'(x) = \frac{kg(x)}{1 + x} \qquad -1 < x < 1$$

(b) Let $h(x) = (1 + x)^{-k} g(x)$ and show that $h'(x) = 0$.

(c) Deduce that $g(x) = (1 + x)^k$.

76. In Exercise 53 in Section 10.2 it was shown that the length of the ellipse $x = a \sin \theta$, $y = b \cos \theta$, where $a > b > 0$, is

$$L = 4a \int_0^{\pi/2} \sqrt{1 - e^2 \sin^2 \theta}\, d\theta$$

where $e = \sqrt{a^2 - b^2}/a$ is the eccentricity of the ellipse.

Expand the integrand as a binomial series and use the result of Exercise 50 in Section 7.1 to express L as a series in powers of the eccentricity up to the term in e^6.