Solutions of Assignment # 9.

Problem 1. Is the following (improper) integral convergent?

a. $\displaystyle\int_1^\infty \frac{\arctan x}{x^2}\,dx$ **b.** $\displaystyle\int_0^1 \frac{\sin x}{x^2}\,dx$

Solution.
a. Since for $x \geq 1$

$$0 \leq \frac{\arctan x}{x^2} \leq \frac{\pi}{2x^2}$$

and

$$\int_1^\infty \frac{1}{x^2}\,dx = -\frac{1}{x}\Big|_1^\infty = -\lim_{x\to\infty}\left(\frac{1}{x}-1\right) = 1,$$

by the Comparison Theorem we obtain that

$$\int_1^\infty \frac{\arctan x}{x^2}\,dx$$

is convergent. Moreover,

$$\int_1^\infty \frac{\arctan x}{x^2}\,dx \leq \frac{\pi}{2}.$$

b. Note

$$\lim_{x\to 0}\frac{\sin x}{x} = 1$$

It implies that there exists $\delta \in (0,1)$ such that for every $x \in (0,\delta)$

$$\frac{\sin x}{x} \geq \frac{1}{2}$$

(indeed, apply definition of the limit with $\varepsilon = 1/2$). Therefore, for every $x \in (0,\delta)$

$$\frac{\sin x}{x^2} \geq \frac{1}{2x}.$$

Now, define $g(x) = \frac{1}{2x}$ on $(0,\delta)$ and $g(x) = 0$ otherwise. We have for every $x \in (0,1]$

$$\frac{\sin x}{x^2} \geq g(x)$$

and

$$\int_0^1 g(x)\,dx = \int_0^\delta \frac{1}{2x}\,dx = \frac{1}{2}\ \ln x\Big|_0^\delta = \lim_{x\to 0+}\frac{\ln\delta - \ln x}{2} = \infty.$$

By the Comparison Theorem, we obtain that

$$\int_0^1 \frac{\sin x}{x^2} = \infty.$$

Answer.
a. convergent **b.** divergent

Problem 2. Find the following integral (or explain why it does not exist)

a. $\displaystyle\int_0^1 \ln x\, dx$ **b.** $\displaystyle\int_0^1 \frac{1}{\sqrt{1-x^2}}\, dx$ **c.** $\displaystyle\int_{-1}^1 \frac{1}{x^2}\, dx$

Solution.
a. Using integration by parts with $u = \ln x$, $dv = dx$ we obtain

$$\int \ln x\ dx = x \ln x - x + C.$$

Thus

$$\int_0^1 \ln x\ dx = (x\ln x - x)\Big|_0^1 = \lim_{x\to 0^+} (\ln 1 - 1 - x\ln x) = -1 - \lim_{x\to 0^+} (x \ln x) = -1$$

(we used here Problem 4c from H/A 6, where applying L'Hospital Rule we proved that $x\ln x \to 0$ as $x \to 0^+$.)
b.

$$\int_0^1 \frac{1}{\sqrt{1-x^2}}\ dx = \arcsin x\Big|_0^1 = \arcsin 1 - \arcsin 0 = \frac{\pi}{2}.$$

Note that in fact we deal here with improper integral and that we are using that arcsin is a continuous function. Namely,

$$\int_0^1 \frac{1}{\sqrt{1-x^2}}\ dx = \lim_{t\to 1}\int_0^t \frac{1}{\sqrt{1-x^2}}\ dx = \lim_{t\to 1} \arcsin x\Big|_0^t = \arcsin x\Big|_0^1.$$

c. We have

$$\int_0^1 \frac{1}{x^2}\ dx = -\frac{1}{x}\Big|_0^1 = \lim_{t\to 0^+}\left(-1 + \frac{1}{x}\right) = \infty$$

and

$$\int_{-1}^0 \frac{1}{x^2}\ dx = -\frac{1}{x}\Big|_{-1}^0 = \lim_{t\to 0^-}\left(-\frac{1}{x} + 1\right) = \infty.$$

Thus

$$\int_{-1}^1 \frac{1}{x^2}\ dx = \infty.$$

□

Answer.

$$\int_0^1 \ln x\ dx = -1, \qquad \int_0^1 \frac{1}{\sqrt{1-x^2}}\ dx = \frac{\pi}{2}, \qquad \int_{-1}^1 \frac{1}{x^2}\ dx = \infty.$$

Problem 3. Let $a \in \mathbb{R}$. Let f, g be non-negative functions defined on $[a, \infty)$. Assume that for every $b \in [a, \infty)$ the functions f, g are bounded and integrable on $[a, b]$. Assume also that

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L < \infty \qquad \text{and} \qquad \int_a^\infty g(x)\ dx \text{ is convergent.}$$

Prove that

$$\int_a^\infty f(x)\ dx \text{ is convergent.}$$

Proof. First note that given $b > a$,

$$\int_a^\infty g(x)\ dx \ \text{ is convergent.}$$

if and only if

$$\int_b^\infty g(x)\ dx \ \text{ is convergent.}$$

Moreover

$$\int_b^\infty g(x)\ dx \le \int_a^\infty g(x)\ dx.$$

Indeed, g is non-negative, so $\int_a^b g(x)\ dx \ge 0$ and

$$\int_a^\infty g(x)\ dx = \int_a^b g(x)\ dx + \int_b^\infty g(x)\ dx.$$

Now, since

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L < \infty$$

we observe that there exists $M \ge a$ such that for every $x \ge M$ one has $f(x)/g(x) \le L+1$ (indeed, apply definition of the limit with $\varepsilon = 1$). Denote

$$I_1 = \int_a^M f(x)\ dx, \qquad I_2 = \int_M^\infty g(x)\ dx, \qquad I_3 = \int_a^\infty g(x)\ dx.$$

It is given that f is integrable on $[a, M]$, so $I_1 < \infty$. We have also observed $I_2 \le I_3 < \infty$.

By the Comparison Theorem we observe

$$\int_a^\infty f(x)\ dx = \int_a^M f(x)\ dx + \int_M^\infty f(x)\ dx \le I_1 + \int_M^\infty (L+1)g(x)\ dx = I_1 + (L+1)I_2 < \infty.$$

It means that

$$\int_a^\infty f(x)\ dx$$

is convergent. $\square$