Solutions of Assignment # 8.

Problem 1. Write out the form of partial fraction decomposition. Do not determine the numerical value of constants.

$$\textbf{a.}\quad \frac{x^3+5x+1}{x^4+4x^3+4x^2}, \qquad\qquad \textbf{b.}\quad \frac{1}{(x^2-2x+1)^2(x^2-x+1)^3}.$$

Solution.

a. Since $x^4+4x^3+4x^2 = x^2(x^2+4x+4) = x^2(x+2)^2$ we have

$$\frac{x^3+5x+1}{x^4+4x^3+4x^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+2} + \frac{D}{(x+2)^2}.$$

b.

$$\frac{1}{(x^2-2x+1)^2(x^2-x+1)^3} = \frac{1}{(x-1)^4(x^2-x+1)^3}$$

$$= \frac{A_1}{x-1} + \frac{A_2}{(x-1)^2} + \frac{A_3}{(x-1)^3} + \frac{A_4}{(x-1)^4} + \frac{C_1x+D_1}{x^2-x+1} + \frac{C_2x+D_2}{(x^2-x+1)^2} + \frac{C_3x+D_3}{(x^2-x+1)^3}$$

Problem 2. Integrate

$$\textbf{a.}\quad \frac{2x^2-x+4}{x^3+4x}, \qquad\qquad \textbf{b.}\quad \frac{x^4+x^3+2x^2+2x+1}{x^3+x^2}.$$

Solution.

a. Since $x^3+4x = x(x^2+4)$,

$$\frac{2x^2-x+4}{x^3+4x} = \frac{A}{x} + \frac{Bx+C}{x^2+4}.$$

It implies $2x^2-x+4 = Ax^2+4A+Bx^2+Cx$. Thus

$$\begin{cases} 2 = A+B \\ -1 = C \\ 4 = 4A \end{cases} \qquad \text{which implies} \qquad \begin{cases} B = 1 \\ C = -1 \\ A = 1 \end{cases}$$

Now,

$$\int \frac{x}{x^2+4}\,dx = [u = x^2+4, \quad du = 2xdx] = \int \frac{dx}{2u} = \frac{1}{2}\ \ln|u| + C_0 = \frac{1}{2}\ln(x^2+4) + C_0.$$

Thus

$$\int \frac{2x^2-x+4}{x^3+4x}\,dx = \int \frac{1}{x}\,dx + \int \frac{x}{x^2+4}\,dx + \int \frac{1}{x^2+4}\,dx = \ln|x| + \frac{1}{2}\ \ln(x^2+4) + \frac{1}{2}\arctan\frac{x}{2} + C.$$

b.

$$\frac{2x^4+2x^3+2x^2+2x+1}{x^3+x^2} = \frac{2x^4+2x^3}{x^3+x^2} + \frac{2x^2+2x+1}{x^2(x+1)} = 2x + \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}.$$

Hence, $2x^2+2x+1 = Ax(x+1)+B(x+1)+Cx^2$. Taking $x=0$ we obtain $B=1$. Taking $x=-1$ we obtain $C=1$. Comparing coefficients in front of x^2 we observe $1=A$. Thus

$$\int \frac{2x^4+2x^3+2x^2+2x+1}{x^3+x^2}\,dx = \int\left(2x + \frac{1}{x} + \frac{1}{x^2} + \frac{1}{x+1}\right) = x^2 + \ln|x| - \frac{1}{x} + \ln|x+1| + C.$$

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