Solutions of Assignment # 7.

Problem 1. Evaluate the following integral

a. $\displaystyle\int t^2\, e^t\, dt,$ **b.** $\displaystyle\int t\, e^{t^2}\, dt,$ **c.** $\displaystyle\int \sin(\ln x)\, dx,$

d. $\displaystyle\int \sin^5 x\, \cos^2 x\, dx,$ **e.** $\displaystyle\int (6x^2{+}2)\cosh(x^3{+}x{+}1)\, dx,$

f. $\displaystyle\int \frac{x}{\sqrt{3-2x-x^2}}\, dx,$ **g.** $\displaystyle\int \frac{1}{x^2\sqrt{x^2+4}}\, dx.$

Solution.

a.

$$\int t^2\, e^t\, dt = \left[\begin{array}{ll} u = t^2 & du = 2t\, dt \\ dv = e^t\, dt & v = e^t \end{array}\right] = t^2\, e^t - 2\int t\, e^t\, dt = \left[\begin{array}{ll} u = t & du = dt \\ dv = e^t\, dt & v = e^t \end{array}\right]$$

$$= t^2\, e^t - 2t\, e^t + 2\int e^t\, dt = t^2\, e^t - 2t\, e^t + 2e^t + C.$$

b.

$$\int t\, e^{t^2}\, dt = [u = t^2, \quad du = 2t\, dt] = \int \frac{e^u\, du}{2} = \frac{e^u}{2} + C = \frac{1}{2}\, e^{t^2} + C.$$

c. First we use substitution $t = \ln x$, so $dt = \frac{1}{x}dx = \frac{1}{e^t}\, dx$. Another way to see this is to write $x = e^t$, so $dx = e^t dt$. Then

$$\int \sin(\ln x)\, dx = \int e^t \sin t\, dt = \left[\begin{array}{ll} u = e^t & du = e^t\, dt \\ dv = \sin t\, dt & v = -\cos t \end{array}\right] = -e^t\cos t + \int e^t \cos t\, dt$$

$$= \left[\begin{array}{ll} u = e^t & du = e^t\, dt \\ dv = \cos t\, dt & v = \sin t \end{array}\right] = -e^t\cos t + e^t \sin t - \int e^t \sin t\, dt.$$

It implies

$$2\int e^t \sin t\, dt = e^t(\sin t - \cos t) + C_0.$$

Thus

$$\int \sin(\ln x)\, dx = \int e^t \sin t\, dt = \frac{e^t}{2}\,(\sin t - \cos t) + C = \frac{x}{2}\,(\sin\ln x - \cos\ln x) + C$$

(here $C = C_0/2$).

d.

$$\int \sin^5 x\, \cos^2 x\, dx = \int \sin x(1-\cos^2 x)^2\, \cos^2 x\, dx = [u = \cos x, \quad du = -\sin x\, dx]$$

$$= -\int (1{-}u^2)^2\, u^2\, du = -\int (u^2{-}2u^4{+}u^6)du = -\frac{u^3}{3} + \frac{2u^5}{5} - \frac{u^7}{7} + C = -\frac{\cos^3 x}{3} + \frac{2\cos^5 x}{5} - \frac{\cos^7 x}{7} + C.$$

e.

$$\int (6x^2+2)\cosh(x^3+x+1)\ dx = [u = x^3+x+1, \quad du = (3x^2+1)dx]$$

$$= 2\int \cosh u\ du = 2\sinh u + C = 2\sinh(x^3+x+1)+C.$$

f.

$$\int \frac{x}{\sqrt{3-2x-x^2}}\ dx = \int \frac{x+1}{\sqrt{3-2x-x^2}}\ dx - \int \frac{1}{\sqrt{3-2x-x^2}}\ dx.$$

First,

$$\int \frac{x+1}{\sqrt{3-2x-x^2}}\ dx = [u = 3-2x-x^2,\ du = (-2-2x)dx = -2(x+1)dx]$$

$$= -\frac{1}{2}\int \frac{du}{\sqrt{u}} = -\sqrt{u}+C_1 = -\sqrt{3-2x-x^2}+C_1.$$

Now note $3-2x-x^2 = 4-(1+x)^2$. We use substitution $1+x = 2\sin t$, then $dx = 2\cos t\ dt$.

$$\int \frac{1}{\sqrt{3-2x-x^2}}\ dx = \int \frac{2\cos t}{2\cos t}\ dt = t + C_1 = \arcsin\frac{1+x}{2} + C_1.$$

Thus

$$\int \frac{x}{\sqrt{3-2x-x^2}}\ dx = -\sqrt{3-2x-x^2}\ -\ \arcsin\frac{1+x}{2}+C$$

(here $C = C_1 + C_2$)).

g. We use substitution $x = 2\tan t$ for $t \in (-\pi/2, \pi/2)$. Then

$$dx = 2\sec^2 t\ dt \quad \text{and} \quad x^2+4 = 4(\tan^2 t+1) = 4\sec^2 t.$$

Thus

$$\int \frac{1}{x^2\sqrt{x^2+4}}\ dx = \int \frac{2\sec^2 t}{(4\tan^2 t)(2\sec t)}\ dt = \int \frac{\cos t}{4\sin^2 t}\ dt = [u = \sin t \quad du = \cos t\, dt]$$

$$= \int \frac{du}{4u^2} = -\frac{1}{4u}+C = -\frac{1}{4\sin t}+C.$$

Now,

$$x^2+4 = 4(\tan^2 t+1) = 4\sec^2 t, \quad \text{so} \quad \cos^2 t = \frac{4}{x^2+4}.$$

Hence $\sin^2 t = \frac{x^2}{x^2+4}$. Finally, since functions

$$x = x(t) = 2\tan t \quad \text{and} \quad \sin t$$

have the same sign on $(-\pi/2, \pi/2)$, we observe $\sin t = \frac{x}{\sqrt{x^2+4}}$. Therefore,

$$\int \frac{1}{x^2\sqrt{x^2+4}}\ dx = -\frac{\sqrt{x^2+4}}{4x}+C.$$