Solutions of Assignment # 6.

Problem 1. Show that for every $x > 0$ one has $(1+x)^{1/x} < e$.

Solution. Since $\ln x$ is a strictly increasing we have for every $x > 0$

$$(1+x)^{1/x} < e \;\Leftrightarrow\; \frac{1}{x}\ \ln(1+x) < \ln e = 1 \;\Leftrightarrow\; \ln(1+x) < x \;\Leftrightarrow\; x - \ln(1+x) > 0.$$

Consider the function $f(x) = x - \ln(1+x)$. Then f is continuous on $[0, \infty)$ and

$$f'(x) = 1 - \frac{1}{x+1} = \frac{x}{x+1} > 0$$

for every $x > 0$. Thus f is strictly increasing and therefore

$$f(x) > f(0) = 0$$

for every $x > 0$. This concludes the proof. $\square$

Problem 2. Show that for every $x > 1$ one has $\ln(x + \sqrt{x^2-1}) = -\ln(x - \sqrt{x^2-1})$.

Solution. Using properties of logarithmic function we have

$$\ln(x+\sqrt{x^2-1}) + \ln(x-\sqrt{x^2-1}) = \ln((x+\sqrt{x^2-1})(x-\sqrt{x^2-1})) = \ln(x^2-(x^2-1)) = \ln 1 = 0,$$

which proves the equality. $\square$

Problem 3. Differentiate

a. $(\ln x)^{\cos x}$ **b.** $(\sin x)^{\ln x}$

Solution. Either using logarithmic differentiation (and Chain rule) saying that

$$(\ln F)' = \frac{F'}{F},$$

or the presentation

$$F = e^{\ln F}$$

(and again Chain rule), we observe

$$F' = F(\ln F)'.$$

Applying that to $F = f^g$ we have $(f^g)' = f^g(g \ln f)'$.

a.

$$\frac{d}{dx}\ (\ln x)^{\cos x} = (\ln x)^{\cos x}\ (\cos x\ \ln\ln x)' = (\ln x)^{\cos x}\ \left(-\sin x\ \ln\ln x + \cos x\ \frac{1}{\ln x}\ \frac{1}{x}\right).$$

b.

$$\frac{d}{dx}\ (\sin x)^{\ln x} = (\sin x)^{\ln x}\ (\ln x\ \ln\sin x)' = (\sin x)^{\ln x}\ \left(\frac{1}{x}\ \ln\sin x + \ln x\ \frac{1}{\sin x}\ \cos x\right).$$

$\square$

Problem 4. Find the following limits (if exist).

a. $\lim_{x\to 0} (\cos x)^{1/\sin^2 x}$ **b.** $\lim_{x\to\infty} (x)^{1/\ln x}$ **c.** $\lim_{x\to 0^+} (x)^x$.

Solution. In this solution we use the following notation $\exp x = e^x$.
a. Take logarithm of the expression and note that we have an indeterminate form of type $\frac{0}{0}$. Then, applying L'Hospital Rule we observe

$$\lim_{x\to 0} \ln (\cos x)^{1/\sin^2 x} = \lim_{x\to 0} \frac{\ln\cos x}{\sin^2 x} = \lim_{x\to 0} \frac{(\ln\cos x)'}{(\sin^2 x)'} = \lim_{x\to 0} \frac{-\sin x}{\cos x \; 2\sin x \;\cos x} = -\lim_{x\to 0} \frac{1}{2\cos^2 x} = -\frac{1}{2}.$$

Since e^x is a continues function and $e^{\ln F} = F$ we obtain

$$\lim_{x\to 0} (\cos x)^{1/\sin^2 x} = \lim_{x\to 0} \exp\left(\ln (\cos x)^{1/\sin^2 x}\right) = \exp\left(\lim_{x\to 0} \ln (\cos x)^{1/\sin^2 x}\right) = e^{-1/2}.$$

b. Note that for every $x > 0$ we have

$$x^{1/\ln x} = \exp\left(\ln x^{1/\ln x}\right) = \exp\left(\frac{\ln x}{\ln x}\right) = e.$$

Thus

$$\lim_{x\to\infty} (x)^{1/\ln x} = e.$$

c. Take logarithm of the expression and note that we have an indeterminate form of type $0\cdot\infty$. Presenting it as ratio (passing to an indeterminate form of type $\frac{0}{0}$) and applying L'Hospital Rule we observe

$$\lim_{x\to 0^+} \ln (x)^x = \lim_{x\to 0^+} x \;\ln x = \lim_{x\to 0^+} \frac{\ln x}{x^{-1}} = \lim_{x\to 0^+} \frac{(\ln x)'}{(x^{-1})'} = \lim_{x\to 0^+} \frac{1}{-x \; x^{-2}} = -\lim_{x\to 0^+} x = 0.$$

Since e^x is a continues function and $e^{\ln F} = F$ we obtain

$$\lim_{x\to 0^+} (x)^x = \lim_{x\to 0^+} \exp\left(\ln (x)^x\right) = \exp\left(\lim_{x\to 0^+} \ln (x)^x\right) = e^0 = 1.$$

$\square$

Answer.

a. $\lim_{x\to 0} (\cos x)^{1/\sin^2 x} = 1/\sqrt{e}$ **b.** $\lim_{x\to\infty} (x)^{1/\ln x} = e$ **c.** $\lim_{x\to 0^+} (x)^x = 1.$