Solutions of Assignment # 5.

Problem 1. Differentiate

a. $\int_{-10}^{x} e^t\,dt$, **b.** $\int_{2}^{x^2} \ln t\,dt$, **c.** $\int_{\sin x}^{4} t^2\,dt$, **d.** $\int_{\tan x}^{\frac{1}{x}} \frac{1}{t}\,dt.$

Solution.

a. $\frac{d}{dx}\int_{-10}^{x} e^t\,dt = e^x.$

b. $\frac{d}{dx}\int_{2}^{x^2} \ln t\,dt = \ln x^2\ (x^2)' = (2\ln x)(2x) = 4x\ln x.$

c. $\frac{d}{dx}\int_{\sin x}^{4} t^2\,dt = -\frac{d}{dx}\int_{4}^{\sin x} t^2\,dt = -(\sin x)^2(\sin x)' = -(\sin x)^2\cos x.$

d. $\frac{d}{dx}\int_{\tan x}^{\frac{1}{x}} \frac{1}{t}\,dt = \frac{d}{dx}\left(\int_{1}^{\frac{1}{x}} \frac{1}{t}\,dt - \int_{1}^{\tan x} \frac{1}{t}\,dt\right) = x\left(\frac{1}{x}\right)' - \frac{(\tan x)'}{\tan x} = -\frac{1}{x} - \frac{1}{\sin x\cos x}.$

$\square$

Problem 2. Let f be a bounded integrable non-negative function on $[a,b]$. Is it true that

a. $\int_a^b f(x)\,dx = 0$ implies $f(x) = 0$ for every x.

b. $\int_a^b f(x)\,dx = 0$ and f is continuous on $[a,b]$ implies $f(x) = 0$ for every x.

Solution.

a. NO. For example let $c \in [a,b]$ (any c works for our solution) and consider function f defined by $f(x) = 0$ for $x \neq c$ and $f(c) = 1$. Then it is easy to see that

$$\int_a^b f(x)\,dx = 0$$

(we did similar examples in class), but f is not identically zero ($f(c) = 1$).

b. YES. Assume NO, that is assume that there exists a continuous non-negative function f on $[a,b]$ such that $\int_a^b f(x)\,dx = 0$ and $f(c) > 0$ for some $c \in [a,b]$. Let $\varepsilon = f(c)/2 > 0$. Since f is continuous, there exists $\delta > 0$ such that for every $x \in [c-\delta, c+\delta]$ one has $f(x) > \varepsilon$ (if $c = a$ or $c = b$ we will take the $[a, a+\delta]$ or $[b-\delta, b]$). Then, since $f \geq 0$ we have

$$0 = \int_a^b f(x)\,dx = \int_a^{c-\delta} f(x)\,dx + \int_{c-\delta}^{c+\delta} f(x)\,dx + \int_{c+\delta}^{b} f(x)\,dx \geq \int_{c-\delta}^{c+\delta} f(x)\,dx \geq 2\delta\,\varepsilon > 0.$$

Contradiction. $\square$

Problem 3. Assume that $\lim_{x\to a} f(x) = L > 0$ and $\lim_{x\to a} g(x) = M$. Prove that $\lim_{x\to a} f(x)^{g(x)} = L^M$.

Solution. The main idea is to use the presentation

$$f^g = e^{\ln(f^g)} = e^{g\ln f}.$$

First note that applying the definition of limit with $\varepsilon = L/2$ we obtain that there exists δ such that for every $x \in [c-\delta, c+\delta]$ one has $f(x) > L/2 > 0$. Thus on this interval $f(x)^{g(x)}$ and $\ln f(x)$ are well-defined. Now, using continuity of logarithm we observe

$$\lim_{x\to a}(\ln f(x)) = \ln L.$$

Therefore,

$$\lim_{x\to a}(\ln f(x)^{g(x)}) = \lim_{x\to a}(g(x)\ln f(x)) = \left(\lim_{x\to a} g(x)\right)\left(\lim_{x\to a}\ln f(x)\right) = M\ln L = \ln L^M.$$

By continuity of e^x we have

$$\lim_{x\to a} f(x)^{g(x)} = \lim_{x\to a} e^{\ln f(x)^{g(x)}} = e^{\ln L^M} = L^M.$$

$\square$

Problem 4. Find domains of the following functions.

a. $f(x) = \log_2(x-3)+\log_7(5-x)$, **b.** $g(x) = \log_2\log_3\log_4 x$,

c. $h(x) = \left(\log_{\sqrt{3}}\tan x\right)^{\pi}$.

Solution.
a. Since $\log_a b$ is defined for $a>0, a\neq 1, b>0$ the domain of f is defined by $x>3$ and $x<5$, that is $x\in(3,5)$.
b. We have to require $x>0$, $\log_4 x>0$, and $\log_3\log_4 x>0$. Equivalently, $x>0$, $x>1$, $\log_4 x>1$. Equivalently, $x>1$ and $x>4$. That is $x>4$.
c. First, from the definition of log we need $\tan x>0$. Since a^{π} is defined for $a>0$ only, we also need $\log_{\sqrt{3}}\tan x>0$, that is $\tan x>1$. Thus we need $x\in(\pi n+\pi/4, \pi n+\pi/2)$, for $n\in\mathbb{Z}$. $\square$

Answer.

$$\mathrm{Dom} f = (3,5), \quad \mathrm{Dom} g = (4,\infty), \quad \mathrm{Dom} h = \bigcup_{n\in\mathbb{Z}}(\pi n+\pi/4, \pi n+\pi/2).$$

Problem 5. Let $a>0$, $x\in\mathbb{R}$. Prove that

$$a^{-x} = \frac{1}{a^x}.$$

Solution. We proved in class that a^x is a continuous function. Take $\{p_n\}_{n=1}^{\infty}$ such that $p_n\in\mathbb{Q}$ for every n and $p_n\to x$ as $n\to\infty$. Then $-p_n\to -x$, and by continuity,

$$a^{-x} = \lim_{n\to\infty} a^{-p_n} = \lim_{n\to\infty}\frac{1}{a^{p_n}} = \frac{1}{\lim_{n\to\infty} a^{p_n}} = \frac{1}{a^x}.$$

(we used also that the equality holds for rational numbers). $\square$

Remark. There is another way to solve Problem 5, namely using definitions. Recall that for $a>1$ we defined a^x as

$$a^x = \sup\{a^p \mid p\in\mathbb{Q},\ p<x\} = \inf\{a^q \mid q\in\mathbb{Q},\ q>x\}$$

(we proved the second equality in the class). Thus we have

$$a^{-x} = \sup\{a^p \mid p \in \mathbb{Q},\ p < -x\} = \sup\{a^p \mid p \in \mathbb{Q},\ -p > x\}.$$

Now note, whenever $p \in \mathbb{Q}$ then $-p \in \mathbb{Q}$, moreover for every $q \in \mathbb{Q}$ there exists $p \in \mathbb{Q}$ such that $q = -p$. In other words, to take supremum over $p \in \mathbb{Q}$, $-p > x$ is the same as to take take supremum over $q \in \mathbb{Q}$, $q > x$ (in fact we use change of variable $q = -p$: when p runs over $\mathbb{Q}$, q runs over $\mathbb{Q}$ as well). Thus, writing $q = -p$, we have

$$a^{-x} = \sup\{a^p \mid p \in \mathbb{Q},\ -p > x\} = \sup\{a^{-q} \mid q \in \mathbb{Q},\ q > x\}.$$

Now we apply property $a^{-q} = 1/a^q$, known for rational numbers q, and properties of supremum and infimum

$$a^{-x} = \sup\{a^{-q} \mid q \in \mathbb{Q},\ q > x\} = \sup\left\{\frac{1}{a^q} \mid q \in \mathbb{Q},\ q > x\right\} = \frac{1}{\inf\{a^q \mid q \in \mathbb{Q},\ q > x\}} = \frac{1}{a^x}.$$

It proves the equality for $a > 1$. The remaining part (for $a \in (0,1]$) is simple.