Solutions of Assignment # 4.

Problem 1. Prove that an increasing function on a closed interval is integrable.

Solution. Let f be an increasing function on an interval $[a, b]$. For every $n \in \mathbb{N}$ we choose partition P_n of $[a, b]$ into intervals of the same length $\ell_n = (b-a)/n$, that is

$$x_0 = a,\ x_1 = x_0 + \ell_n = a + \ell_n,\ \ldots,\ x_i = x_{i-1} + \ell_n = a + i\ell_n,\ \ldots,\ x_n = a + n\ell_n = b.$$

Since f is an increasing, we have for every i

$$m_i = \inf_{[x_{i-1},x_i]} f = f(x_{i-1}), \quad M_i = \sup_{[x_{i-1},x_i]} f = f(x_i).$$

Therefore, using $|x_i - x_{i-1}| = \ell_n$, we observe

$$\mathcal{U}(P_n, f) - \mathcal{L}(P_n, f) = \sum_{i=0}^{n} M_i(x_i - x_{i-1}) - \sum_{i=0}^{n} m_i(x_i - x_{i-1}) = \ell_n \sum_{i=0}^{n} (M_i - m_i)$$

$$= \ell_n \sum_{i=0}^{n} (f(x_i) - f(x_{i-1})) = \ell_n(f(b) - f(a)) = \frac{1}{n}\,(b-a)(f(b) - f(a)) \to 0 \ \text{ as } \ n \to \infty.$$

Thus for every $\varepsilon > 0$ there exists a partition P (namely, P_n with n satisfying $n > (b-a)(f(b) - f(a))/\varepsilon$) such that

$$\mathcal{U}(P, f) - \mathcal{L}(P, f) < \varepsilon.$$

By the Cauchy criterion this implies that f is integrable. □

Remark. The proof for a decreasing function repeats the same lines (or one can consider the function $g = -f$, which will be increasing, so as we just proved integrable, and apply corresponding theorem saying that if g is integrable then so is $-g$).

Problem 2. Prove that every Lipschitz function is uniformly continuous.

Solution. Let $f : A \to \mathbb{R}$ be a Lipschitz function, that is, there exists $C \in \mathbb{R}$ such that for every $x, y \in \mathrm{Dom} f$ one has

$$|f(x) - f(y)| \le C|x - y|.$$

Fix an arbitrary $\varepsilon > 0$. Choose $\delta = \varepsilon/C$. Then for every $x, y \in \mathrm{Dom} f$ with $|x - y| < \delta$ we have

$$|f(x) - f(y)| \le C|x - y| < C\delta = \varepsilon.$$

This proves the statement. □

Problem 3. Is the following function uniformly continuous?

a. $f(x) = \sqrt{x}$ on $[1, \infty)$ **b.** $f(x) = \sqrt{x}$ on $[0, \infty)$.

Solution. We show that the answer is YES for both questions (in fact we need to show it only for the second question, however the proof in the first case is much simpler). First note that for every $x, y \in \mathbb{R}$ satisfying $0 \le y \le x$ we have

$$|f(x) - f(y)| = \sqrt{x} - \sqrt{y} = \frac{x-y}{\sqrt{x} + \sqrt{y}} \le \frac{x-y}{\sqrt{x}}.$$

Thus, if, in addition, $x \geq 1$ we have

$$|f(x) - f(y)| \leq |x - y|.$$

It shows that f is Lipschitz on $[1, \infty)$ and, by Problem 2, gives answer YES to the first problem (but not only, note we have this inequality for every $x \geq 1$ and every $0 \leq y \leq x$).

Now we fix an arbitrary $\varepsilon > 0$. Since f is continuous, by the Cantor Theorem, it is uniformly continuous on $[0, 1]$. Thus for every ε there exists $\delta_0 > 0$ such that for every $x, y \in [0, 1]$ with $|x - y| < \delta_0$ we have $|f(x) - f(y)| < \varepsilon$.

Now we choose $\delta = \min\{\varepsilon, \delta_0\}$. For every $0 \leq y \leq x$ with $|x - y| < \delta$ we have

Case 1. if $x \geq 1$ then

$$|f(x) - f(y)| \leq |x - y| < \delta \leq \varepsilon;$$

Case 2. if $x \leq 1$ then (since $\delta \leq \delta_0$)

$$|f(x) - f(y)| < \varepsilon.$$

This proves the desired result. □

Remark 1. Of course we can prove this result without using the Cantor Theorem, namely, just by providing formula for $\delta(\varepsilon)$ (**Exer.**).

Remark 2. If you want to argue like "f is uniformly continuous on $[0, 1]$ and on $[1, \infty)$, so f is uniformly continuous on the union" then first you have to prove such a general statement.

Problem 4. Is the following function a Lipschitz function?

a. $f(x) = x^2$ on $[-1, 1]$ **b.** $f(x) = x^2$ on $[1, \infty)$.

c. $f(x) = \sqrt{x}$ on $[1, \infty)$ **d.** $f(x) = \sqrt{x}$ on $[0, 1]$.

Solution.

a. YES. Indeed, for every $x, y \in [-1, 1]$ one has

$$|f(x) - f(y)| = |x^2 - y^2| = |x - y|\ |x + y| \leq |x - y|\ (|x| + |y|) \leq 2\ |x - y|.$$

b. NO. Indeed, assume yes. Then there exists a constant $C > 1$ (if $C_0 < 1$ works then any $C > 1$ works as well) such that for every $x, y \geq 1$

$$|f(x) - f(y)| = |x^2 - y^2| \leq C|x - y|.$$

Take $x = C$ and $y = 2C$. Then $|x^2 - y^2| = 3C^2$, $|x - y| = C$. Hence

$$3C^2 \leq C^2.$$

Contradiction.

c. YES. Indeed, for every $x, y \in [1, \infty)$ one has

$$|f(x) - f(y)| = |\sqrt{x} - \sqrt{y}| = \frac{|x - y|}{\sqrt{x} + \sqrt{y}} \leq |x - y|.$$

d. NO. Indeed, assume yes. Then there exists a constant $C > 1$ (if $C_0 < 1$ works then any $C > 1$ works as well) such that for every $x, y \in [0, 1]$

$$|f(x) - f(y)| = |\sqrt{x} - \sqrt{y}| \leq C|x - y|.$$

Take $x = 0$ and $y = 1/(2C)^2$. Then $|\sqrt{x} - \sqrt{y}| = 1/(2C)$ and $|x - y| = 1/(2C)^2$. Hence

$$\frac{1}{2C} \leq \frac{1}{4C}.$$

Contradiction. □