Solutions of Assignment # 3.

Problem 1. Find the following limits (if exist).

a. $\lim_{x\to 0} \frac{x}{\arcsin x}$, **b.** $\lim_{x\to 0} \frac{x}{\arccos x}$, **c.** $\lim_{x\to 0^+} \frac{\sin x - x}{x^4}$,

d. $\lim_{x\to \pi} \frac{\sin x}{x}$, **e.** $\lim_{x\to 2\pi} \frac{\sin x}{x-2\pi}$, **f.** $\lim_{x\to\infty} (x\, \text{arccot} x)$,

Solution. Note, in first 5 problems we deal with ratio of continuous functions, so to it is easy to calculate limits of nominator and denominator – just take the value of the function at the corresponding point. Moreover, functions are differentiable and, thus, we can apply L'Hospital's Rule whenever we have indeterminate forms.

a. Here we have indeterminate form of the type $\frac{0}{0}$. Thus

$$\lim_{x\to 0} \frac{x}{\arcsin x} = \lim_{x\to 0} \frac{(x)'}{(\arcsin x)'} = \lim_{x\to 0} \frac{1}{1/\sqrt{1-x^2}} = 1$$

b. Since $\arccos 0 = \pi/2$, we don't have indeterminate form here

$$\lim_{x\to 0} \frac{x}{\arccos x} = \frac{0}{\arccos 0} = 0.$$

c. Here we initially have indeterminate form of the type $\frac{0}{0}$ as well as after applying L'Hospital's Rule twice. So we apply it three times. Note that each time, before applying it, we check the conditions.

$$\lim_{x\to 0^+} \frac{\sin x - x}{x^4} = \lim_{x\to 0^+} \frac{\cos x - 1}{4x^3} = \lim_{x\to 0^+} \frac{-\sin x}{12x^2} = \lim_{x\to 0^+} \frac{-\cos x}{24x} = -\infty.$$

In the last equality we used that

$$\lim_{x\to 0^+} (-\cos x) = -1 < 0, \quad \lim_{x\to 0^+} x = 0, \quad \text{and that} \quad x > 0.$$

d. We don't have indeterminate form here

$$\lim_{x\to \pi} \frac{\sin x}{x} = \frac{\sin \pi}{\pi} = 0.$$

e. Here we have indeterminate form of the type $\frac{0}{0}$

$$\lim_{x\to 2\pi} \frac{\sin x}{x - 2\pi} = \lim_{x\to 2\pi} \frac{\cos x}{1} = 1.$$

f. Since

$$\lim_{x\to\infty} \text{arccot} x = 0,$$

we have indeterminate form of the type $0 \cdot \infty$. So we present it as indeterminate form of the type $\frac{0}{0}$ and apply L'Hospital's Rule.

$$\lim_{x\to\infty} (x\, \text{arccot} x) = \lim_{x\to\infty} \frac{\text{arccot} x}{1/x} = \lim_{x\to\infty} \frac{-1/(1+x^2)}{-1/x^2} = \lim_{x\to\infty} \frac{x^2}{1+x^2} = 1.$$

$\square$

Problem 2. Differentiate (simplify your answer, where possible)

a. $\arcsin(\sin x)$ on $(0, \pi/2)$, **b.** $\arcsin(\sin x)$ on $(\pi/2, \pi)$, **c.** $\arctan(\cos^2 x)$.

Solution.
a. For $x \in (0, \pi/2)$ we have $\arcsin(\sin x) = x$, therefore

$$(\arcsin(\sin x))' = (x)' = 1$$

(we used also that we work on an open interval).

b. Note that, by the definition, $\arcsin x$ is the inverse of $\sin x$ restricted to $[-\pi/2, \pi/2]$. Thus, $\arcsin : [-1, 1] \to [-\pi/2, \pi/2]$. In particular, $\arcsin x \le \pi/2$, so $\arcsin(\sin x) \ne x$ for $x > \pi/2$. We use here that for every x one has $\sin x = \sin(\pi - x)$. Note, if $x \in (\pi/2, \pi)$ then $\pi - x \in (0, \pi/2)$, which implies

$$\arcsin(\sin x) = \arcsin(\sin(\pi - x)) = \pi - x$$

for $x \in (\pi/2, \pi)$. Thus, for $x \in (\pi/2, \pi)$

$$(\arcsin(\sin x))' = (\pi - x)' = -1$$

(we used again that we work on an open interval).

c. Applying Chain Rule we have

$$\left(\arctan(\cos^2 x)\right)' = \frac{1}{1 + \cos^4 x} \, 2\cos x \, (-\sin x) = -\frac{2\cos x \sin x}{1 + \cos^4 x} = -\frac{\sin(2x)}{1 + \cos^4 x}.$$

$\square$

Remark. Note, we can apply Chain Rule in first two questions as well:

$$(\arcsin(\sin x))' = \frac{1}{\sqrt{1 - \sin^2 x}} \cos x.$$

Then, we have to simplify the expression, using that

$$\sqrt{1 - \sin^2 x} = \cos x \quad \text{on} \quad (0, \pi/2) \qquad \text{and} \qquad \sqrt{1 - \sin^2 x} = -\cos x \quad \text{on} \quad (0, \pi/2).$$

Problem 3. Prove that for every x one has $\arctan x + \mathrm{arccot} x = \frac{\pi}{2}$.

Solution. Denote $f(x) = \arctan x + \mathrm{arccot} x$. Then

$$f'(x) = \frac{1}{1 + x^2} + \left(-\frac{1}{1 + x^2}\right) = 0$$

on $\mathbb{R}$. It implies that f is a constant function, that is, there exists a constant C such that for every x one has $f(x) = C$. To compute C we evaluate f at 0:

$$C = f(0) = \arctan 0 + \mathrm{arccot} 0 = 0 + \frac{\pi}{2} = \frac{\pi}{2}.$$

This proves the result. $\square$

Problem 4. Find **a.** $\arcsin(\sin\pi)$ **b.** $\arccos(\cos(-\pi))$.

Solution. **a.** $\arcsin(\sin\pi) = \arcsin 0 = 0$ **b.** $\arccos(\cos(-\pi)) = \arccos(-1) = \pi$. $\square$

Problem 5. Find the inverse function of

a. $f(x) = \dfrac{1}{x}$ **b.** $g(x) = \sqrt{1-x^2}$ **c.** $h(x) = (x+1)^3$.

Solution. Note, $F(x) = y$ iff $x = F^{-1}(y)$. Thus, to find inverse we have to solve equation $F(x) = y$ (i.e. to find a formula for x in terms of y.)

a.

$$f(x) = y \;\Leftrightarrow\; \frac{1}{x} = y \;\Leftrightarrow\; x = \frac{1}{y}.$$

Thus,

$$f^{-1}(x) = \frac{1}{x}.$$

b. First note that the domain of g is $[-1,1]$ and g is not one-to-one on its domain (since, for example, $g(-1) = g(1) = 0$). Hence g is not invertible. Thus the answer to the problem (as is posted): the inverse does not exists.

However, if we change the domain to $[0,1]$, say, then g becomes invertible. Indeed, under assumption that $x, y \geq 0$ (note here that $g \geq 0$) we have

$$g(x) = y \;\Leftrightarrow\; 1 - x^2 = y^2 \;\Leftrightarrow\; x^2 = 1 - y^2 \;\Leftrightarrow\; x = \sqrt{1-y^2}.$$

Thus,

$$g^{-1}(y) = \sqrt{1-y^2}.$$

Another way to get the invertible function is to restrict g to the interval $[-1,0]$. We denote such a function by $\bar{g}$ to distinguish from the previous case. Then, under assumption that $x \leq 0$, $y \geq 0$ (note here that $\bar{g} \geq 0$) we have

$$\bar{g}(x) = y \;\Leftrightarrow\; 1 - x^2 = y^2 \;\Leftrightarrow\; x^2 = 1 - y^2 \;\Leftrightarrow\; x = -\sqrt{1-y^2}.$$

Thus,

$$\bar{g}^{-1}(y) = -\sqrt{1-y^2}.$$

Note that we got different formulae for g^{-1} and $\bar{g}^{-1}$.

c.

$$h(x) = y \;\Leftrightarrow\; (x+1)^3 = y \;\Leftrightarrow\; x + 1 = y^{1/3} \;\Leftrightarrow\; y^{1/3} - 1.$$

Thus,

$$h^{-1}(y) = y^{1/3} - 1.$$

$\square$

Answer.

a. $f^{-1}(x) = \dfrac{1}{x}$ **b.** $g^{-1}(y) = \sqrt{1-x^2}$ **c.** $h^{-1}(x) = x^{1/3} - 1$.