Solutions of Assignment # 2.

Problem 2. Show that the equation $x^2 = x \sin x + \cos x$ has exactly two real roots.

Solution. Consider function $f(x) = x^2 - x\sin x - \cos x$. We have $f'(x) = 2x - \sin x - x\cos x + \sin x = x(2 - \cos x)$. Thus $f' > 0$ on $(0, \infty)$ and $f' \leq 0$ on $(-\infty, 0)$. It means that f is strictly increasing on $(0, \infty)$ and strictly decreasing on $(-\infty, 0)$.

Note that $f(0) = -1$ and $f(2) = 4 - 2\sin x - \cos x > 0$. Since f is continuous and $f(0) < 0 < f(2)$, there exists a positive a such that $f(a) = 0$ (moreover, $a \in (0, 2)$). On the other hand we cannot have more than one positive number a such that $f(a) = 0$ (indeed, if there are $0 < a_1 < a_2$ such that $f(a_1) = f(a_2) = 0$ then f is NOT strictly increasing on $(0, \infty)$). It shows that there exists exactly one positive number a satisfying $f(a) = 0$.

Similarly, there exists exactly one negative number b satisfying $f(b) = 0$. It proves the result. $\square$