Solutions of Assignment # 11.

Problem 1. Find all $p > 0$ such that the following series is convergent.

$$\textbf{a.} \quad \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p} \qquad\qquad \textbf{b.} \quad \sum_{n=2}^{\infty} \frac{(-1)^n}{n(\ln n)^p}$$

Solution.
a. We use the integral tests. Note that the function $f(x) = \frac{1}{x(\ln x)^p}$ is positive and decreasing on $[2, \infty)$. Using substitution $u = \ln x$ the corresponding integral can be evaluated. For $p > 1$

$$\int_2^{\infty} f(x)\, dx = \lim_{t\to\infty} \frac{(\ln x)^{1-p}}{1-p}\Big|_2^t = \lim_{t\to\infty} \left(\frac{(\ln t)^{1-p}}{1-p} - \frac{(\ln 2)^{1-p}}{1-p} \right) = \frac{(\ln 2)^{1-p}}{p-1} < \infty.$$

It implies that the series is convergent.
For $p = 1$

$$\int_2^{\infty} f(x)\, dx = \lim_{t\to\infty} (\ln(\ln x))\Big|_2^t = \lim_{t\to\infty} (\ln(\ln t) - \ln(\ln 2)) = \infty.$$

It implies that the series is divergent.
For $p < 1$

$$\int_2^{\infty} f(x)\, dx = \lim_{t\to\infty} \frac{(\ln x)^{1-p}}{1-p}\Big|_2^t = \lim_{t\to\infty} \left(\frac{(\ln t)^{1-p}}{1-p} - \frac{(\ln 2)^{1-p}}{1-p} \right) = \infty.$$

It implies that the series is divergent.

b. Here we have an alternating series. Note that

$$\left\{ \frac{1}{n(\ln n)^p} \right\}_{n=1}^{\infty}$$

is decreasing and tends to 0 as $n \to \infty$. Therefore, by the Altering Test, the series is convergent. □

Answer. **a.** The series is convergent for $p > 1$ (divergent for $p \in (0, 1)$. **b.** The series is convergent for every $p > 0$.

Problem 2. Does the following series converge?

$$\textbf{a.} \quad \sum_{n=1}^{\infty} (-1)^n \ \ln \frac{2n^2}{(n+10)^2} \qquad\qquad \textbf{b.} \quad \sum_{n=1}^{\infty} n^{-1/2} \ \tan\left(\frac{1}{\sqrt{n}}\right)$$

$$\textbf{c.} \quad \sum_{n=1}^{\infty} \left(\frac{\arctan n}{\pi}\right)^n \qquad\qquad \textbf{d.} \quad \sum_{n=1}^{\infty} \frac{n!}{n^n}$$

Solution.
a. The series is divergent, since the general term does not tend to 0, indeed

$$\lim_{n\to\infty} \ln \frac{2n^2}{(n+10)^2} = \ln \lim_{n\to\infty} \frac{2n^2}{n^2 + 20n + 100} = \ln 2 > 0.$$

b. We use the limit comparison test.

$$\lim_{n\to\infty} \frac{n^{-1/2}\ \tan\left(\frac{1}{\sqrt{n}}\right)}{1/n} = \lim_{z=\frac{1}{\sqrt{n}}\to 0} \frac{\tan z}{z} = 1.$$

Since

$$\sum_{n=1}^{\infty} \frac{1}{n} = \infty$$

we obtain that our series is also divergent.

c. Here we use the Root Test. Since

$$\lim_{n\to\infty} \frac{\arctan n}{\pi} = \frac{1}{2} < 1,$$

we obtain that our series is convergent.

d.
Way 1. Assume that $k = [n/2]$, that is either $n = 2k$ or $n = 2k + 1$. Clearly, $k \leq n/2$ and for every $n \geq 2$ we have $k \geq n/4$ (indeed $k \geq (n-1)/2 \geq n/4$). We have

$$n! = 1 \cdot 2 \cdot 3\ \ldots \cdot k \cdot (k+1) \cdot \ldots \cdot n \leq k^k n^{n-k}.$$

Thus, for $n \geq 2$,

$$\frac{n!}{n^n} \leq \frac{k^k}{n^k} \leq \left(\frac{k}{n}\right)^k \leq \left(\frac{1}{2}\right)^{n/4} = \left(2^{-1/4}\right)^n.$$

Since $2^{-1/4} < 1$, it implies that

$$\sum_{n=2}^{\infty} \frac{n!}{n^n} \leq \sum_{n=2}^{\infty} \left(2^{-1/4}\right)^n < \infty,$$

so our series is also convergent. **Way 2.** We apply Ratio test. Let $a_n = n!/n^n$. Then

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!n^n}{n!(n+1)^{(}n+1)} = \frac{n^n}{(n+1)^n} = \left(1 + \frac{1}{n}\right)^{-n} \to \frac{1}{e} < 1 \qquad \text{as } n \to \infty.$$

□

Answer. **a.** Divergent. **b.** Divergent. **c.** Convergent. **d.** Convergent.