Solutions of Assignment # 10.

Problem 1. Evaluate the following integral

a. $\displaystyle\int_2^\infty \frac{dx}{\ln x}$, **b.** $\displaystyle\int_0^{\pi/2} \tan x \; dx$, **c.** $\displaystyle\int_2^\infty \frac{dx}{x\ln x}$,

d. $\displaystyle\int_1^2 \frac{dx}{x\ln x}$, **e.** $\displaystyle\int_2^\infty \frac{dx}{x\ln^2 x}$, **f.** $\displaystyle\int_1^2 \frac{dx}{x\ln^2 x}$.

Solution.
a. Note that $\ln x \le x$ for every $x \ge 1$ (in fact for every $x > 0$). Indeed, consider $f(x) = x - \ln x$. Then $f'(x) = 1 - 1/x \ge 0$, so f is an increasing function. Therefore for every $x \ge 1$ we have $f(x) \ge f(1) = 1 - \ln 1 = 1$. It implies $x \ge \ln x$. Thus for every $x \ge 2$ one has

$$\frac{1}{\ln x} \ge \frac{1}{x}.$$

Note

$$\int_2^\infty \frac{1}{x}\; dx = \lim_{t\to\infty} \ln x\Big|_2^t = \lim_{t\to\infty}(\ln t - \ln 2) = \infty.$$

By the comparison theorem it implies

$$\int_2^\infty \frac{1}{\ln x}\; dx = \infty.$$

b. Using substitution $u = \cos x$, $du = \sin x\; dx$ we obtain

$$\int \tan x\; dx = \int \frac{\sin x}{\cos x}\; dx = \int \frac{-du}{u} = -\ln|u| + C = -\ln|\cos x| + C.$$

Therefore,

$$\int_0^{\pi/2} \tan x\; dx = \lim_{t\to\left(\frac{\pi}{2}\right)^-} (-\ln|\cos x|)\Big|_0^t = \lim_{t\to\left(\frac{\pi}{2}\right)^-} (-\ln|\cos t| + \ln 1) = \infty$$

(we used that $\cos t \to 0$ as $t \to \pi/2$ and $\ln z \to -\infty$ as $z \to 0$).

c. Using substitution $u = \ln x$, $du = \frac{1}{x}\; dx$ we obtain

$$\int \frac{dx}{x\ln x} = \int \frac{du}{u} = \ln|u| + C = \ln|\ln x| + C.$$

Therefore,

$$\int_2^\infty \frac{dx}{x\ln x}\; dx = \lim_{t\to\infty} (\ln|\ln x|)\Big|_2^t = \lim_{t\to\infty} (\ln|\ln t| - \ln(\ln 2)) = \infty$$

(we used that $\ln t \to \infty$ as $t \to \infty$).

d. Using the indefinite integral from the previous problem we have

$$\int_1^2 \frac{dx}{x\ln x}\,dx = \lim_{t\to 1^+} \left(\ln|\ln x|\right)\Big|_t^2 = \lim_{t\to 1^+} \left(\ln(\ln 2) - \ln|\ln t|\right) = \infty$$

(we used that $\ln t \to 0$ as $t \to 1^+$, $\ln t > 0$ for $t > 1$, and $\ln z \to -\infty$ as $z \to 0^+$).

e. Using substitution $u = \ln x$, $du = \frac{1}{x}\,dx$ we obtain

$$\int \frac{dx}{x\ln^2 x} = \int \frac{du}{u^2} = -\frac{1}{u} + C = -\frac{1}{\ln x} + C.$$

Therefore,

$$\int_2^\infty \frac{dx}{x\ln x}\,dx = \lim_{t\to\infty} \left(-\frac{1}{\ln x}\right)\Big|_2^t = \lim_{t\to\infty} \left(-\frac{1}{\ln t} + \frac{1}{\ln 2}\right) = \frac{1}{\ln 2}$$

(we used that $\ln t \to \infty$ as $t \to \infty$, so $\frac{1}{\ln t} \to 0$ as $t \to \infty$).

f. Using the indefinite integral from the previous problem we have

$$\int_1^2 \frac{dx}{x\ln x}\,dx = \lim_{t\to 1^+} \left(-\frac{1}{\ln x}\right)\Big|_t^2 = \lim_{t\to 1^+} \left(-\frac{1}{\ln 2} + \frac{1}{\ln t}\right) = \infty$$

(we used that $\ln t \to 0$ as $t \to 1^+$ and $\ln t > 0$ for $t > 1$, so $\frac{1}{\ln t} \to \infty$ as $t \to 1^+$).

Answer.

a. $\displaystyle\int_2^\infty \frac{dx}{\ln x} = \infty$, **b.** $\displaystyle\int_0^{\pi/2} \tan x\ dx = \infty$, **c.** $\displaystyle\int_2^\infty \frac{dx}{x\ln x} = \infty$,

d. $\displaystyle\int_1^2 \frac{dx}{x\ln x} = \infty$, **e.** $\displaystyle\int_2^\infty \frac{dx}{x\ln^2 x} = \frac{1}{\ln 2}$, **f.** $\displaystyle\int_1^2 \frac{dx}{x\ln^2 x} = \infty$.