Solutions of Assignment # 1.

Problem 1. Does $f'(a)$ exists? Explain your answer.

a. $a = 0, \quad f(x) = \begin{cases} 0, & \text{if } x \geq 0, \\ x^2, & \text{if } x < 0; \end{cases}$

b. $a = 0, \quad f(x) = \begin{cases} -x, & \text{if } x \geq 0, \\ x^2, & \text{if } x < 0; \end{cases}$

c. $a = 0, \quad f(x) = \begin{cases} -x^2, & \text{if } x \geq 0, \\ x^2, & \text{if } x < 0; \end{cases}$

d. $a = 0, \quad f(x) = \begin{cases} \sin x, & \text{if } x \geq 0, \\ x, & \text{if } x < 0; \end{cases}$

e. $a = 2, \quad f(x) = \begin{cases} x+1, & \text{if } x \geq 0, \\ x-1, & \text{if } x < 0; \end{cases}$

f. $a = 1, \quad f(x) = \begin{cases} 1-x, & \text{if } x \geq 0, \\ (1-x)(2-x), & \text{if } x < 0; \end{cases}$

Solution.

a.

$$f'_+(0) = \lim_{x\to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0^+} \frac{0}{x} = 0,$$

$$f'_-(0) = \lim_{x\to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0^-} \frac{x^2}{x} = \lim_{x\to 0^-} x = 0.$$

Since $f'_+(0) = f'_-(0) = 0$ we obtain $f'(0) = 0$.

b.

$$f'_+(0) = \lim_{x\to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0^+} \frac{-x}{x} = -1,$$

$$f'_-(0) = \lim_{x\to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0^-} \frac{x^2}{x} = \lim_{x\to 0^-} x = 0.$$

Since $f'_+(0) \neq f'_-(0) = 0$ we obtain that $f'(0)$ does not exist.

c.

$$f'_+(0) = \lim_{x\to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0^+} \frac{-x^2}{x} = \lim_{x\to 0^+} (-x) = 0,$$

$$f'_-(0) = \lim_{x\to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0^-} \frac{x^2}{x} = \lim_{x\to 0^-} x = 0.$$

Since $f'_+(0) = f'_-(0) = 0$ we obtain $f'(0) = 0$.

d.

$$f'_+(0) = \lim_{x\to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0^+} \frac{\sin x}{x} = 1,$$

$$f'_-(0) = \lim_{x\to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0^-} \frac{x}{x} = 1.$$

Since $f'_+(0) = f'_-(0) = 1$ we obtain $f'(0) = 1$.

e. By the definition of the limit it is enough to consider f on the interval $(1,3)$ only. On this interval $f(x) = x+1$, so $f'(x) = 1$ on $(1,3)$. In particular $f'(2) = 1$.

f. By the definition of the limit it is enough to consider f on the interval $(0,1)$ only. On this interval $f(x) = 1-x$, so $f'(x) = -1$ on $(1,3)$. In particular $f'(1) = -1$. □

Answer.
a. $f'(0) = 0$ **b.** $f'(0)$ DNE (does not exist) **c.** $f'(0) = 0$
d. $f'(0) = 1$ **e.** $f'(0) = 1$ **f.** $f'(0) = -1$

Remark 1. Sometimes it is possible to avoid the direct application of the definition, but one should be very careful justifying the solution. For example, another way to solve Problem 1a is:
Consider function $g(x) = -x^2$. It is differentiable on $\mathbb{R}$ and coincides with f on $(-\infty, 0]$. It implies (can you prove such a general statement?) that $g'(x) = f'(x)$ on $(-\infty, 0)$ and moreover $g'(0) = f'_+(0)$. Since $g'(x) = 2x$, $f'_+(0) = g'(0) = 0$. Similarly, considering $h(x) = 0$, which coincides with f on $(0, \infty)$, we obtain $f'_-(0) = h'(0) = 0$.

Remark (exer.) 2. Do Problem 1 for

$$a = 2, \quad f(x) = \begin{cases} x+1, & \text{if } x \geq 2, \\ x-1, & \text{if } x < 2; \end{cases}$$

$$a = 1, \quad f(x) = \begin{cases} 1-x, & \text{if } x \geq 1, \\ (1-x)(2-x), & \text{if } x < 1; \end{cases}$$

Problem 2. Differentiate (don't simplify)

a. $\sin(x^2+2x+1)$, **b.** $\cot(1/x)$, **c.** $\dfrac{x^2}{x+1}$, **d.** $\sqrt{x+\sqrt{x}}$.

Answers.

a.

$$f'(x) = (2x+2)\cos\left(x^2+2x+1\right)$$

b.

$$f'(x) = \frac{1}{x^2}\,\frac{1}{\sin^2(1/x)}$$

c.

$$f'(x) = \frac{2x(x+1) - x^2}{(x+1)^2}$$

d.

$$f'(x) = \frac{1}{2\sqrt{x+\sqrt{x}}}\,\frac{1}{2\sqrt{x}}.$$

□

Problem 3. Using Lagrange Mean Value Theorem prove that for every $a, b \in \mathbb{R}$ one has $|\sin a - \sin b| \leq |a - b|$.

Solution. If $a = b$ then both sides equal to 0 and the inequality is trivial. Hence we assume $a \neq b$.

Since $|a-b| = |b-a|$ and $|\sin a - \sin b| = |\sin b - \sin a|$ we may assume that $a < b$ (otherwise we interchange the role of a and b). Now we consider $f(x) = \sin x$ on $[a, b]$. As we know from the class, f is continuous on $[a, b]$ and differentiable on (a, b). Therefore we can apply Lagrange MVT and obtain that there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{\sin a - \sin b}{b - a}.$$

On the other hand we know that $f'(x) = \cos x$, so

$$\left| \frac{\sin a - \sin b}{b - a} \right| = |\cos c| \leq 1.$$

This implies the desired result. □

Problem 4. A number a is called a fixed point of a function f if $f(a) = a$. Assuming that f is differentiable and that for every x one has $f'(x) \neq 1$ prove that f has at most one fixed point.

Solution. Assume that we have two such numbers, say $a < b$. Then $f(a) = a$, $f(b) = b$, f is differentiable (in particular on (a, b)). Since f is differentiable it is continuous (in particular on $[a, b]$). By Lagrange MVT there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{\sin a - \sin b}{b - a} = \frac{b - a}{b - a} = 1,$$

which contradicts the assumption $f' \neq 1$. □

Problem 5. Using differentiation prove that for every $x \in \mathbb{R}$ one has

$$\cos x \geq 1 - \frac{x^2}{2}.$$

Solution. Consider the function f defined by $f(x) = \cos x - (1 - \frac{x^2}{2})$. Then f is differentiable everywhere and

$$f'(x) = -\sin x + x.$$

As we know $x \geq \sin x$ for non-negative x. Thus $f' \geq 0$ on $[0, \infty)$, so f is increasing on $[0, \infty)$. Thus, for every $x \geq 0$ we have $f(x) \geq f(0) = 0$. If $x \leq 0$ we have $f' \leq 0$ (indeed, let $y = -x \geq 0$, so $y \geq \sin y$, hence $-x \geq \sin(-x) = -\sin x$.) It means that f is decreasing on $(-\infty, 0]$. Thus, for every $x \leq 0$ we have $f(x) \geq f(0) = 0$. This implies that for every x we have $f(x) = \cos x - (1 - \frac{x^2}{2}) \geq 0$, which proves the result. □

Remark (exer). Prove the following inequalities, using the same method:

$$(1 + x)^n \geq 1 + nx \quad \text{for every} \ \ n \in \mathbb{N} \ \text{ and every } x \geq -1;$$

$$x + \frac{1}{x} \geq 2 \quad \text{for every} \ \ x > 0.$$