Quiz # 6

Problem 2. Is the following series convergent?

a. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{1}{n}\right)$ **b.** $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n+2}{10n+1}$ **c.** $\displaystyle\sum_{n=1}^{\infty} (-1)^n \ln\frac{n}{n+1}$

Solution.
a. Since $\sin x < x$ for every $x > 0$, by comparison theorem we have

$$\sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{1}{n}\right) \le \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty.$$

Therefore the series is convergent. (we could use also that

$$\lim_{n\to\infty} \frac{(1/n)\sin(1/n)}{1/n^2} = 1$$

and again that $\sum(1/n^2)$ is convergent.)

b. Note that

$$\lim_{n\to\infty} \left|(-1)^n \frac{n+2}{10n+1}\right| = \frac{1}{10} \neq 0,$$

therefore the series is divergent.

c. We have an alternating series. Denote $c_n = \ln(n/(n+1))$. Then

$$\lim_{n\to\infty} \left|(-1)^n c_n\right| = \ln 1 = 0$$

and

$$\left|(-1)^n c_n\right| = \ln\frac{n+1}{n} = \ln\left(1+\frac{1}{n}\right)$$

is decreasing (since ln is an increasing function). Therefore, by an alternating series test, we obtain that the series is convergent. □

Answer. **a.** The series is convergent. **b.** The series is divergent. **c.** The series is convergent.