Quiz # 5

Problem 1. Write out the form of partial fraction decomposition.

$$\frac{x^4+3x^2+2}{x^3(x^2-1)(x^2+1)^2}$$

Solution. Since $x^2-1=(x-1)(x+1)$ we have

$$\frac{x^4+3x^2+2}{x^3(x^2-1)(x^2+1)^2}=\frac{A_1}{x}+\frac{A_2}{x^2}+\frac{A_3}{x^3}+\frac{B_1}{x-1}+\frac{B_2}{x+1}+\frac{C_1x+D_1}{x^2+1}+\frac{C_2x+D_2}{(x^2+1)^2}.$$

Problem 2. Find **a.** $\displaystyle\int_0^{\pi/2}\frac{\cos x}{x}\,dx$, **b.** $\displaystyle\int_0^1 x\ln x\,dx$.

Solution.
a. Note, $\cos x\geq 0$ on $[0,\pi/2]$ and $\cos x\geq 1/2$ on $[0,\pi/3]$. Setting $f(x)=\frac{1}{2x}$ for $x\in[0,\pi/3]$ and $f(x)=0$ for $x\in[\pi/3,\pi/2]$, we observe

$$\frac{\cos x}{x}\geq f(x) \text{ on } (0,\pi/2] \quad \text{and} \quad \int_0^{\pi/2} f(x)\,dx=\int_0^{\pi/3}\frac{1}{2x}\,dx=\frac{1}{2}\lim_{t\to 0^+}\ln x\Big|_t^{\frac{\pi}{2}}=\frac{1}{2}\lim_{t\to 0^+}(\ln\frac{\pi}{2}-\ln t)=\infty.$$

By comparison theorem we obtain

$$\int_0^{\pi/2}\frac{\cos x}{x}\,dx=\infty.$$

b. Note

$$\int x\ln x\,dx=\begin{bmatrix} u=\ln x & du=(1/x)dx\\ dv=x\,dx & v=x^2/2\end{bmatrix}=\frac{x^2}{2}\ln x-\int\frac{x}{2}\,dx=\frac{x^2}{2}\ln x-\frac{x^2}{4}+C.$$

Thus

$$\int_0^1 x\ln x\,dx=\lim_{t\to 0^+}\left(\frac{x^2}{2}\ln x-\frac{x^2}{4}\right)\Big|_t^1=\lim_{t\to 0^+}\left(\frac{1}{2}\ln 1-\frac{1}{4}-\frac{t^2}{2}\ln t+\frac{t^2}{4}\right)=-\frac{1}{4}$$

(we used here that $t^2\ln t\to 0$ as $t\to 0^+$, which can be proved either using L'Hospital Rule or an example in the class, saying $t\ln t\to 0$ as $t\to 0^+$). $\square$

Answer.

a. $\displaystyle\int_0^{\pi/2}\frac{\cos x}{x}\,dx=\infty$ **b.** $\displaystyle\int_0^1 x\ln x\,dx=-\frac{1}{4}.$

Remark. The choice of $\pi/3$ in Problem 2a is not important. Indeed, we can choose (and fix) any $s\in(0,\pi/2)$. Then $\cos x\geq\cos s>0$ on $[0,s]$, so

$$\int_0^{\pi/2}\frac{\cos x}{x}\,dx\geq\int_0^{s}\frac{\cos x}{x}\,dx\geq\int_0^{s}\frac{\cos s}{x}\,dx=\cos s\,\ln x\Big|_{x=0}^{x=s}=\lim_{t\to 0^+}(\cos s\,\ln s-\cos s\,\ln t)=\infty.$$