Quiz # 1

Problem 1. State the Rolle Theorem.

Theorem. Let $a < b$, f be a continuous function on [a, b]. Assume that f is differentiable on (a, b) and that $f(a) = f(b)$. Then there exists $c \in (a, b)$ such that $f'(c) = 0$.

Problem 2. Differentiate following functions (do not simplify).

a. $f(x) = \dfrac{\cos x}{x}$ **b.** $g(x) = \sin(x^4)$ **c.** $h(x) = \begin{cases} \sqrt{x}, & \text{if } x > 0, \\ \tan x, & \text{if } x \leq 0; \end{cases}$

Solution.

a. $$f'(x) = \frac{-x\sin x - \cos x}{x^2}, \qquad \text{where we used that} \quad \left(\frac{F}{G}\right)' = \frac{F'G - FG'}{G^2}.$$

b. $g'(x) = 4x^3\cos(x^4)$, where we used the chain rule $(F(G(x)))' = F'(G(x))G'(x)$.

c. First notice that on an open interval $(0, \infty)$ we have $h(x) = \sqrt{x}$, therefore on this interval $h'(x) = (\sqrt{x})' = \frac{1}{2\sqrt{x}}$. Similarly, on an open interval $(-\infty, 0)$ we have $h(x) = \tan x$, therefore on this interval $h'(x) = (\tan x)' = \cos^{-2} x$. At point 0 the function change the behavior, so we have to check the corresponding limits. Note, $h(0) = \tan 0 = 0$. Thus

$$\lim_{x \to 0^+} \frac{h(x) - h(0)}{x - 0} = \lim_{x \to 0^+} \frac{\sqrt{x}}{x} = \lim_{x \to 0^+} \frac{1}{\sqrt{x}} = \infty.$$

This means that $h'_+(0)$ does not exists (derivative or one-sided derivative exists if the corresponding limit exists as a number). Therefore, $h'(0)$ does not exists. So we obtained

$$h(x) = \begin{cases} \frac{1}{2\sqrt{x}}, & \text{if } x > 0, \\ \frac{1}{\cos^2 x}, & \text{if } x < 0; \end{cases} \qquad \text{and} \qquad h' \text{ DNE at } 0.$$

□

Remark. Note that

$$h'_-(0) = \lim_{x \to 0^-} \frac{h(x) - h(0)}{x - 0} = \lim_{x \to 0^-} \frac{\tan x}{x} = \lim_{x \to 0^-} \frac{\sin x}{x\cos x} = \lim_{x \to 0^-} \frac{\sin x}{x} \lim_{x \to 0^-} \frac{1}{\cos x} = 1,$$

but it does not imply that $h'(0)$ exists. Note also that you may NOT argue like

$$h'(x) = \frac{1}{\cos^2 x}$$

for $x < 0$ so, passing to limits, $h'_-(0) = 1$, since you are not given that the derivative is continuous (you are not even given that it exists.)

Example. Consider function

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

Then f is continuous on $\mathbb{R}$, f is differentiable on $\mathbb{R}$ (find the derivative!), but f' is not continuous at 0 (prove that!).