Examples to the last lecture

1. $f(x) = e^x$ $\qquad f^{(n)}(x) = e^x \qquad f^{(n)}(0) = 1$

$$T_{f,0}(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad R_N(x) \xrightarrow[N\to\infty]{} 0 \quad (\forall x)$$

Thus $\forall x$ $\qquad e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

2. $f(x) = \sin x \qquad f^{(4k+1)}(x) = \cos x \qquad f^{(4k+2)}(x) = -\sin x$

$f^{(4k+3)}(x) = -\cos x \qquad f^{(4k)}(x) = \sin x$

$\Rightarrow \quad f^{(2k)}(0) = 0, \quad f^{(2k+1)}(0) = (-1)^k$

$$\Rightarrow T_{f,0}(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots =$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} \qquad R_N(x) \xrightarrow[N\to\infty]{} 0 \quad (\forall x)$$

Thus $\forall x$ $\qquad \sin x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$

3. $f(x) = \cos x$ $\qquad$ Similarly,

$$\cos x = 1 - x^2/2 + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$$

4. $f(x) = \frac{1}{1-x}$

We know that $\forall x \in (-1,1)$ $\qquad \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n =$

$= 1 + x + x^2 + x^3 + \dots$

By Lemma 2.1 we obtain that $T_{f,0}(x) = \sum_{n=0}^{\infty} x^n$ (for every $|x|<1$).

We can compute $T_{f,0}$ directly:

$f(x) = f^{(0)}(x) = (1-x)^{-1}$, $f'(x) = (1-x)^{-2}$, $f''(x) = 2(1-x)^{-3}$,

$f'''(x) = 3\cdot 2(1-x)^{-4}$, ..., $f^{(n)}(x) = n!\,(1-x)^{-n-1}$

$\Rightarrow f^{(0)}(0) = 1$, $f'(0) = 1$, ..., $f^{(n)}(0) = n!$

$$\Rightarrow T_{f,0}(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} x^n$$

<u>Remark</u> Using $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ for $|x|<1$ we can obtain other series, e.g.

<u>4a</u> $$\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n$$

for $|-x| < 1 \Leftrightarrow |x| < 1$

<u>4b</u> $$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

for $|-x^2| < 1 \Leftrightarrow |x| < 1$

<u>4c</u> $$\frac{x^5}{1+x^6} = x^5 \cdot \frac{1}{1-(-x)^6} = x^5 \sum_{n=0}^{\infty} (-x^6)^n = \sum_{n=0}^{\infty} (-1)^n x^{6n+5}$$

5. $f(x) = \ln(1+x)$ $\quad f'(x) = \frac{1}{1+x} = (1+x)^{-1}$

$f''(x) = -(1+x)^{-2}$, $f'''(x) = 2(1+x)^{-3}$, ..., $f^{(n)}(x) = (n-1)!\,(-1)^{n+1}(1+x)^{-n}$

$\Rightarrow f(0) = \ln 1 = 0$, $f'(0) = 1$, ..., $f^{(n)}(0) = (n-1)!\,(-1)^{n+1}$

$\Rightarrow T_{f,0}(x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$ Convergent for $|x| < 1$.

Thus $\forall\ |x|<1 \quad \ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$

$\Rightarrow \ln(1-x) = \ln(1-(-x)) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \dots = -\sum_{n=1}^{\infty} \frac{x^n}{n}$

<u>Remark</u> <u>Another proof</u> $\frac{d}{dx} f(x) = \frac{d}{dx} \ln(1+x) = \frac{1}{1+x} =$

$= \sum_{n=0}^{\infty} (-1)^n x^n$. Then, by Th 1.5, integrating,

$\ln(1+x) + C_1 = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} = \sum_{m=1}^{\infty} (-1)^{m+1} \frac{x^m}{m}$

Pluging in $x_0 = 0$ we obtain $C_1 = 0$.

6. $f(x) = \arctan x \qquad f'(x) = \frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}$

By Th 1.5, integrating,

~~[illegible]~~ $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = \arctan x + C$

Pluging in $x = 0$ we obtain $0 = C$, thus

$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$