13. $f(x) = \sqrt[3]{x}$, $[0,1]$. f is continuous on $\mathbb{R}$ and differentiable on $(-\infty, 0) \cup (0, \infty)$, so f is continuous on $[0,1]$ and differentiable on $(0,1)$. $f'(c) = \dfrac{f(b)-f(a)}{b-a} \;\Leftrightarrow\; \dfrac{1}{3c^{2/3}} = \dfrac{f(1)-f(0)}{1-0} \;\Leftrightarrow\; \dfrac{1}{3c^{2/3}} = \dfrac{1-0}{1} \;\Leftrightarrow\; 3c^{2/3} = 1 \;\Leftrightarrow\; c^{2/3} = \frac{1}{3} \;\Leftrightarrow\; c^2 = \left(\frac{1}{3}\right)^3 = \frac{1}{27} \;\Leftrightarrow\; c = \pm\sqrt{\frac{1}{27}} = \pm\frac{\sqrt{3}}{9}$, but only $\frac{\sqrt{3}}{9}$ is in $(0,1)$.

14. $f(x) = \dfrac{x}{x+2}$, $[1,4]$. f is continuous on $[1,4]$ and differentiable on $(1,4)$. $f'(c) = \dfrac{f(b)-f(a)}{b-a} \;\Leftrightarrow\; \dfrac{2}{(c+2)^2} = \dfrac{\frac{2}{3}-\frac{1}{3}}{4-1} \;\Leftrightarrow\; (c+2)^2 = 18 \;\Leftrightarrow\; c = -2 \pm 3\sqrt{2}$. $-2 + 3\sqrt{2} \approx 2.24$ is in $(1,4)$.

15. $f(x) = |x-1|$. $f(3) - f(0) = |3-1| - |0-1| = 1$. Since $f'(c) = -1$ if $c < 1$ and $f'(c) = 1$ if $c > 1$, $f'(c)(3-0) = \pm 3$ and so is never equal to 1. This does not contradict the Mean Value Theorem since $f'(1)$ does not exist.

16. $f(x) = \dfrac{x+1}{x-1}$. $f(2) - f(0) = 3 - (-1) = 4$. $f'(x) = \dfrac{1(x-1) - 1(x+1)}{(x-1)^2} = \dfrac{-2}{(x-1)^2}$. Since $f'(x) < 0$ for all x (except $x = 1$), $f'(c)(2-0)$ is always < 0 and hence cannot equal 4. This does not contradict the Mean Value Theorem since f is not continuous at $x = 1$.

17. Let $f(x) = 1 + 2x + x^3 + 4x^5$. Then $f(-1) = -6 < 0$ and $f(0) = 1 > 0$. Since f is a polynomial, it is continuous, so the Intermediate Value Theorem says that there is a number c between -1 and 0 such that $f(c) = 0$. Thus, the given equation has a real root. Suppose the equation has distinct real roots a and b with $a < b$. Then $f(a) = f(b) = 0$. Since f is a polynomial, it is differentiable on (a,b) and continuous on $[a,b]$. By Rolle's Theorem, there is a number r in (a,b) such that $f'(r) = 0$. But $f'(x) = 2 + 3x^2 + 20x^4 \geq 2$ for all x, so $f'(x)$ can never be 0. This contradiction shows that the equation can't have two distinct real roots. Hence, it has exactly one real root.

18. Let $f(x) = 2x - 1 - \sin x$. Then $f(0) = -1 < 0$ and $f(\pi/2) = \pi - 2 > 0$. f is the sum of the polynomial $2x - 1$ and the scalar multiple $(-1) \cdot \sin x$ of the trigonometric function $\sin x$, so f is continuous (and differentiable) for all x. By the Intermediate Value Theorem, there is a number c in $(0, \pi/2)$ such that $f(c) = 0$. Thus, the given equation has at least one real root. If the equation has distinct real roots a and b with $a < b$, then $f(a) = f(b) = 0$. Since f is continuous on $[a,b]$ and differentiable on (a,b), Rolle's Theorem implies that there is a number r in (a,b) such that $f'(r) = 0$. But $f'(r) = 2 - \cos r > 0$ since $\cos r \leq 1$. This contradiction shows that the given equation can't have two distinct real roots, so it has exactly one real root.

19. Let $f(x) = x^3 - 15x + c$ for x in $[-2,2]$. If f has two real roots a and b in $[-2,2]$, with $a < b$, then $f(a) = f(b) = 0$. Since the polynomial f is continuous on $[a,b]$ and differentiable on (a,b), Rolle's Theorem implies that there is a number r in (a,b) such that $f'(r) = 0$. Now $f'(r) = 3r^2 - 15$. Since r is in (a,b), which is contained in $[-2,2]$, we have $|r| < 2$, so $r^2 < 4$. It follows that $3r^2 - 15 < 3 \cdot 4 - 15 = -3 < 0$. This contradicts $f'(r) = 0$, so the given equation can't have two real roots in $[-2,2]$. Hence, it has at most one real root in $[-2,2]$.

20. $f(x) = x^4 + 4x + c$. Suppose that $f(x) = 0$ has three distinct real roots a, b, d where $a < b < d$. Then $f(a) = f(b) = f(d) = 0$. By Rolle's Theorem there are numbers c_1 and c_2 with $a < c_1 < b$ and $b < c_2 < d$ and $0 = f'(c_1) = f'(c_2)$, so $f'(x) = 0$ must have at least two real solutions. However $0 = f'(x) = 4x^3 + 4 = 4(x^3 + 1) = 4(x+1)(x^2 - x + 1)$ has as its only real solution $x = -1$. Thus, $f(x)$ can have at most two real roots.