34. (a) If $dP/dt = 0$, the population is stable (it is constant).

(b) $\dfrac{dP}{dt} = 0 \;\Rightarrow\; \beta P = r_0\left(1 - \dfrac{P}{P_c}\right)P \;\Rightarrow\; \dfrac{\beta}{r_0} = 1 - \dfrac{P}{P_c} \;\Rightarrow\; \dfrac{P}{P_c} = 1 - \dfrac{\beta}{r_0} \;\Rightarrow\; P = P_c\left(1 - \dfrac{\beta}{r_0}\right).$

If $P_c = 10{,}000$, $r_0 = 5\% = 0.05$, and $\beta = 4\% = 0.04$, then $P = 10{,}000(1 - \frac{4}{5}) = 2000$.

(c) If $\beta = 0.05$, then $P = 10{,}000(1 - \frac{5}{5}) = 0$. There is no stable population.

35. (a) If the populations are stable, then the growth rates are neither positive nor negative; that is,

$\dfrac{dC}{dt} = 0$ and $\dfrac{dW}{dt} = 0$.

(b) "The caribou go extinct" means that the population is zero, or mathematically, $C = 0$.

(c) We have the equations $\dfrac{dC}{dt} = aC - bCW$ and $\dfrac{dW}{dt} = -cW + dCW$. Let $dC/dt = dW/dt = 0$, $a = 0.05$, $b = 0.001$, $c = 0.05$, and $d = 0.0001$ to obtain $0.05C - 0.001CW = 0$ **(1)** and $-0.05W + 0.0001CW = 0$ **(2)**. Adding 10 times **(2)** to **(1)** eliminates the CW-terms and gives us $0.05C - 0.5W = 0 \;\Rightarrow\; C = 10W$. Substituting $C = 10W$ into **(1)** results in $0.05(10W) - 0.001(10W)W = 0 \;\Leftrightarrow\; 0.5W - 0.01W^2 = 0 \;\Leftrightarrow\; 50W - W^2 = 0 \;\Leftrightarrow\; W(50 - W) = 0 \;\Leftrightarrow\; W = 0$ or 50. Since $C = 10W$, $C = 0$ or 500. Thus, the population pairs (C, W) that lead to stable populations are $(0, 0)$ and $(500, 50)$. So it is possible for the two species to live in harmony.

3.5 Derivatives of Trigonometric Functions

1. $f(x) = x - 3\sin x \;\Rightarrow\; f'(x) = 1 - 3\cos x$

2. $f(x) = x\sin x \;\Rightarrow\; f'(x) = x \cdot \cos x + (\sin x) \cdot 1 = x\cos x + \sin x$

3. $y = \sin x + 10\tan x \;\Rightarrow\; y' = \cos x + 10\sec^2 x$

4. $y = 2\csc x + 5\cos x \;\Rightarrow\; y' = -2\csc x\cot x - 5\sin x$

5. $g(t) = t^3\cos t \;\Rightarrow\; g'(t) = t^3(-\sin t) + (\cos t)\cdot 3t^2 = 3t^2\cos t - t^3\sin t$ or $t^2(3\cos t - t\sin t)$

6. $g(t) = 4\sec t + \tan t \;\Rightarrow\; g'(t) = 4\sec t\tan t + \sec^2 t$

7. $h(\theta) = \theta\csc\theta - \cot\theta \;\Rightarrow\; h'(\theta) = \theta(-\csc\theta\cot\theta) + (\csc\theta)\cdot 1 - (-\csc^2\theta) = \csc\theta - \theta\csc\theta\cot\theta + \csc^2\theta$

8. $y = u(a\cos u + b\cot u) \;\Rightarrow$

$y' = u(-a\sin u - b\csc^2 u) + (a\cos u + b\cot u)\cdot 1 = a\cos u + b\cot u - au\sin u - bu\csc^2 u$

9. $y = \dfrac{x}{\cos x} \;\Rightarrow\; y' = \dfrac{(\cos x)(1) - (x)(-\sin x)}{(\cos x)^2} = \dfrac{\cos x + x\sin x}{\cos^2 x}$

10. $y = \dfrac{1 + \sin x}{x + \cos x} \;\Rightarrow$

$$y' = \frac{(x + \cos x)(\cos x) - (1 + \sin x)(1 - \sin x)}{(x + \cos x)^2} = \frac{x\cos x + \cos^2 x - (1 - \sin^2 x)}{(x + \cos x)^2}$$

$$= \frac{x\cos x + \cos^2 x - (\cos^2 x)}{(x + \cos x)^2} = \frac{x\cos x}{(x + \cos x)^2}$$

11. $f(\theta) = \dfrac{\sec\theta}{1 + \sec\theta} \;\Rightarrow$

$$f'(\theta) = \frac{(1 + \sec\theta)(\sec\theta\tan\theta) - (\sec\theta)(\sec\theta\tan\theta)}{(1 + \sec\theta)^2} = \frac{(\sec\theta\tan\theta)[(1 + \sec\theta) - \sec\theta]}{(1 + \sec\theta)^2} = \frac{\sec\theta\tan\theta}{(1 + \sec\theta)^2}$$

12. $y = \dfrac{\tan x - 1}{\sec x} \Rightarrow$

$$\frac{dy}{dx} = \frac{\sec x \sec^2 x - (\tan x - 1)\sec x \tan x}{\sec^2 x} = \frac{\sec x\,(\sec^2 x - \tan^2 x + \tan x)}{\sec^2 x} = \frac{1 + \tan x}{\sec x}$$

Another method: Simplify y first: $y = \sin x - \cos x \Rightarrow y' = \cos x + \sin x$.

13. $y = \dfrac{\sin x}{x^2} \Rightarrow y' = \dfrac{x^2 \cos x - (\sin x)(2x)}{(x^2)^2} = \dfrac{x(x\cos x - 2\sin x)}{x^4} = \dfrac{x\cos x - 2\sin x}{x^3}$

14. $y = \csc\theta\,(\theta + \cot\theta) \Rightarrow$

$$\begin{aligned} y' &= \csc\theta\,(1 - \csc^2\theta) + (\theta + \cot\theta)(-\csc\theta\cot\theta) = \csc\theta\,(1 - \csc^2\theta - \theta\cot\theta - \cot^2\theta) \\ &= \csc\theta\,(-\cot^2\theta - \theta\cot\theta - \cot^2\theta) \qquad [1 + \cot^2\theta = \csc^2\theta] \\ &= \csc\theta\,(-\theta\cot\theta - 2\cot^2\theta) = -\csc\theta\cot\theta\,(\theta + 2\cot\theta) \end{aligned}$$

15. $y = \sec\theta\,\tan\theta \Rightarrow y' = \sec\theta\,(\sec^2\theta) + \tan\theta\,(\sec\theta\tan\theta) = \sec\theta\,(\sec^2\theta + \tan^2\theta)$

Using the identity $1 + \tan^2\theta = \sec^2\theta$, we can write alternative forms of the answer as

$$\sec\theta\,(1 + 2\tan^2\theta) \qquad \text{or} \qquad \sec\theta\,(2\sec^2\theta - 1)$$

16. Recall that if $y = fgh$, then $y' = f'gh + fg'h + fgh'$. $y = x\sin x\cos x \Rightarrow$

$$\frac{dy}{dx} = \sin x\cos x + x\cos x\cos x + x\sin x\,(-\sin x) = \sin x\cos x + x\cos^2 x - x\sin^2 x$$

17. $\dfrac{d}{dx}(\csc x) = \dfrac{d}{dx}\left(\dfrac{1}{\sin x}\right) = \dfrac{(\sin x)(0) - 1(\cos x)}{\sin^2 x} = \dfrac{-\cos x}{\sin^2 x} = -\dfrac{1}{\sin x}\cdot\dfrac{\cos x}{\sin x} = -\csc x\cot x$

18. $\dfrac{d}{dx}(\sec x) = \dfrac{d}{dx}\left(\dfrac{1}{\cos x}\right) = \dfrac{(\cos x)(0) - 1(-\sin x)}{\cos^2 x} = \dfrac{\sin x}{\cos^2 x} = \dfrac{1}{\cos x}\cdot\dfrac{\sin x}{\cos x} = \sec x\tan x$

19. $\dfrac{d}{dx}(\cot x) = \dfrac{d}{dx}\left(\dfrac{\cos x}{\sin x}\right) = \dfrac{(\sin x)(-\sin x) - (\cos x)(\cos x)}{\sin^2 x} = -\dfrac{\sin^2 x + \cos^2 x}{\sin^2 x} = -\dfrac{1}{\sin^2 x} = -\csc^2 x$

20. $f(x) = \cos x \Rightarrow$

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} = \lim_{h\to 0}\frac{\cos(x+h) - \cos x}{h} = \lim_{h\to 0}\frac{\cos x\cos h - \sin x\sin h - \cos x}{h} \\ &= \lim_{h\to 0}\left(\cos x\,\frac{\cos h - 1}{h} - \sin x\,\frac{\sin h}{h}\right) = \cos x\lim_{h\to 0}\frac{\cos h - 1}{h} - \sin x\lim_{h\to 0}\frac{\sin h}{h} \\ &= (\cos x)(0) - (\sin x)(1) = -\sin x \end{aligned}$$

21. $y = \tan x \Rightarrow y' = \sec^2 x \Rightarrow$ the slope of the tangent line at $\left(\frac{\pi}{4}, 1\right)$ is $\sec^2\frac{\pi}{4} = \left(\sqrt{2}\right)^2 = 2$ and an equation of the tangent line is $y - 1 = 2\left(x - \frac{\pi}{4}\right)$ or $y = 2x + 1 - \frac{\pi}{2}$.

22. $y = (1 + x)\cos x \Rightarrow y' = (1 + x)(-\sin x) + \cos x \cdot 1$. At $(0, 1)$, $y' = 1$, and an equation of the tangent line is $y - 1 = 1(x - 0)$, or $y = x + 1$.

23. $y = x + \cos x \Rightarrow y' = 1 - \sin x$. At $(0, 1)$, $y' = 1$, and an equation of the tangent line is $y - 1 = 1(x - 0)$, or $y = x + 1$.

35. $\lim_{x\to 0}\frac{\sin 3x}{x} = \lim_{x\to 0}\frac{3\sin 3x}{3x}$ [multiply numerator and denominator by 3]

$= 3\lim_{3x\to 0}\frac{\sin 3x}{3x}$ [as $x\to 0, 3x\to 0$]

$= 3\lim_{\theta\to 0}\frac{\sin\theta}{\theta}$ [let $\theta = 3x$]

$= 3(1)$ [Equation 2]

$= 3$

36. $$\lim_{x\to 0}\frac{\sin 4x}{\sin 6x} = \lim_{x\to 0}\left(\frac{\sin 4x}{x}\cdot\frac{x}{\sin 6x}\right) = \lim_{x\to 0}\frac{4\sin 4x}{4x}\cdot\lim_{x\to 0}\frac{6x}{6\sin 6x}$$
$$= 4\lim_{x\to 0}\frac{\sin 4x}{4x}\cdot\frac{1}{6}\lim_{x\to 0}\frac{6x}{\sin 6x} = 4(1)\cdot\frac{1}{6}(1) = \frac{2}{3}$$

37. $$\lim_{t\to 0}\frac{\tan 6t}{\sin 2t} = \lim_{t\to 0}\left(\frac{\sin 6t}{t}\cdot\frac{1}{\cos 6t}\cdot\frac{t}{\sin 2t}\right) = \lim_{t\to 0}\frac{6\sin 6t}{6t}\cdot\lim_{t\to 0}\frac{1}{\cos 6t}\cdot\lim_{t\to 0}\frac{2t}{2\sin 2t}$$
$$= 6\lim_{t\to 0}\frac{\sin 6t}{6t}\cdot\lim_{t\to 0}\frac{1}{\cos 6t}\cdot\frac{1}{2}\lim_{t\to 0}\frac{2t}{\sin 2t} = 6(1)\cdot\frac{1}{1}\cdot\frac{1}{2}(1) = 3$$

38. $$\lim_{\theta\to 0}\frac{\cos\theta - 1}{\sin\theta} = \lim_{\theta\to 0}\frac{\dfrac{\cos\theta - 1}{\theta}}{\dfrac{\sin\theta}{\theta}} = \frac{\lim_{\theta\to 0}\dfrac{\cos\theta - 1}{\theta}}{\lim_{\theta\to 0}\dfrac{\sin\theta}{\theta}} = \frac{0}{1} = 0$$

39. $$\lim_{\theta\to 0}\frac{\sin(\cos\theta)}{\sec\theta} = \frac{\sin\left(\lim_{\theta\to 0}\cos\theta\right)}{\lim_{\theta\to 0}\sec\theta} = \frac{\sin 1}{1} = \sin 1$$

40. $$\lim_{t\to 0}\frac{\sin^2 3t}{t^2} = \lim_{t\to 0}\left(\frac{\sin 3t}{t}\cdot\frac{\sin 3t}{t}\right) = \lim_{t\to 0}\frac{\sin 3t}{t}\cdot\lim_{t\to 0}\frac{\sin 3t}{t}$$
$$= \left(\lim_{t\to 0}\frac{\sin 3t}{t}\right)^2 = \left(3\lim_{t\to 0}\frac{\sin 3t}{3t}\right)^2 = (3\cdot 1)^2 = 9$$

41. $$\lim_{x\to 0}\frac{\cot 2x}{\csc x} = \lim_{x\to 0}\frac{\cos 2x\sin x}{\sin 2x} = \lim_{x\to 0}\cos 2x\left[\frac{(\sin x)/x}{(\sin 2x)/x}\right] = \lim_{x\to 0}\cos 2x\left[\frac{\lim_{x\to 0}[(\sin x)/x]}{2\lim_{x\to 0}[(\sin 2x)/2x]}\right]$$
$$= 1\cdot\frac{1}{2\cdot 1} = \frac{1}{2}$$

42. $$\lim_{x\to\pi/4}\frac{\sin x - \cos x}{\cos 2x} = \lim_{x\to\pi/4}\frac{\sin x - \cos x}{\cos^2 x - \sin^2 x} = \lim_{x\to\pi/4}\frac{\sin x - \cos x}{(\cos x + \sin x)(\cos x - \sin x)}$$
$$= \lim_{x\to\pi/4}\frac{-1}{\cos x + \sin x} = \frac{-1}{\cos\frac{\pi}{4} + \sin\frac{\pi}{4}} = -\frac{1}{\sqrt{2}}$$

43. Divide numerator and denominator by θ. ($\sin\theta$ also works.)

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta + \tan\theta} = \lim_{\theta\to 0}\frac{\dfrac{\sin\theta}{\theta}}{1 + \dfrac{\sin\theta}{\theta}\dfrac{1}{\cos\theta}} = \frac{\lim_{\theta\to 0}\dfrac{\sin\theta}{\theta}}{1 + \lim_{\theta\to 0}\dfrac{\sin\theta}{\theta}\lim_{\theta\to 0}\dfrac{1}{\cos\theta}} = \frac{1}{1 + 1\cdot 1} = \frac{1}{2}$$

44. $$\lim_{x\to 1}\frac{\sin(x-1)}{x^2 + x - 2} = \lim_{x\to 1}\frac{\sin(x-1)}{(x+2)(x-1)} = \lim_{x\to 1}\frac{1}{x+2}\lim_{x\to 1}\frac{\sin(x-1)}{x-1} = \tfrac{1}{3}\cdot 1 = \tfrac{1}{3}$$