

5.

$n$	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$(1+x)^{-3}$	1
1	$-3(1+x)^{-4}$	-3
2	$12(1+x)^{-5}$	12
3	$-60(1+x)^{-6}$	-60
4	$360(1+x)^{-7}$	360
$\vdots$	$\vdots$	$\vdots$

$$\begin{aligned}(1+x)^{-3} &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \dots \\ &= 1 - 3x + \frac{4 \cdot 3}{2!}x^2 - \frac{5 \cdot 4 \cdot 3}{3!}x^3 + \frac{6 \cdot 5 \cdot 4 \cdot 3}{4!}x^4 - \dots \\ &= 1 - 3x + \frac{4 \cdot 3 \cdot 2}{2 \cdot 2!}x^2 - \frac{5 \cdot 4 \cdot 3 \cdot 2}{2 \cdot 3!}x^3 + \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2}{2 \cdot 4!}x^4 - \dots \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n (n+2)! x^n}{2(n!)} = \sum_{n=0}^{\infty} \frac{(-1)^n (n+2)(n+1)x^n}{2}\end{aligned}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+3)(n+2)x^{n+1}}{2} \cdot \frac{2}{(n+2)(n+1)x^n} \right| = |x| \lim_{n \rightarrow \infty} \frac{n+3}{n+1} = |x| < 1 \text{ for convergence,}$$

so  $R = 1$  (Ratio Test).

6.

$n$	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\ln(1+x)$	0
1	$(1+x)^{-1}$	1
2	$-(1+x)^{-2}$	-1
3	$2(1+x)^{-3}$	2
4	$-6(1+x)^{-4}$	-6
5	$24(1+x)^{-5}$	24
$\vdots$	$\vdots$	$\vdots$

$$\begin{aligned}\ln(1+x) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 \\ &\quad + \frac{f^{(4)}(0)}{4!}x^4 + \frac{f^{(5)}(0)}{5!}x^5 + \dots \\ &= x - \frac{1}{2}x^2 + \frac{2}{6}x^3 - \frac{6}{24}x^4 + \frac{24}{120}x^5 - \dots \\ &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \dots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n\end{aligned}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{n+1} \cdot \frac{n}{x^n} \right| = \lim_{n \rightarrow \infty} \frac{|x|}{1+1/n} = |x| < 1 \text{ for}$$

convergence, so  $R = 1$ .

7.

$n$	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$e^{5x}$	1
1	$5e^{5x}$	5
2	$5^2 e^{5x}$	25
3	$5^3 e^{5x}$	125
4	$5^4 e^{5x}$	625
$\vdots$	$\vdots$	$\vdots$

$$e^{5x} = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{5^n}{n!} x^n.$$

$$\begin{aligned}\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left[ \frac{5^{n+1} |x|^{n+1}}{(n+1)!} \cdot \frac{n!}{5^n |x|^n} \right] \\ &= \lim_{n \rightarrow \infty} \frac{5|x|}{n+1} = 0 < 1 \text{ for all } x, \text{ so } R = \infty.\end{aligned}$$

8.

$n$	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$xe^x$	0
1	$(x+1)e^x$	1
2	$(x+2)e^x$	2
3	$(x+3)e^x$	3
$\vdots$	$\vdots$	$\vdots$

$$xe^x = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{n}{n!} x^n = \sum_{n=1}^{\infty} \frac{n}{n!} x^n = \sum_{n=1}^{\infty} \frac{x^n}{(n-1)!}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left[ \frac{|x|^{n+1}}{n!} \cdot \frac{(n-1)!}{|x|^n} \right] = \lim_{n \rightarrow \infty} \frac{|x|}{n} = 0 < 1 \text{ for all } x, \text{ so } R = \infty.$$

9.

$n$	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\sinh x$	0
1	$\cosh x$	1
2	$\sinh x$	0
3	$\cosh x$	1
4	$\sinh x$	0
$\vdots$	$\vdots$	$\vdots$

$$f^{(n)}(0) = \begin{cases} 0 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd} \end{cases} \text{ so } \sinh x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$$

Use the Ratio Test to find  $R$ . If  $a_n = \frac{x^{2n+1}}{(2n+1)!}$ , then

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{x^{2n+3}}{(2n+3)!} \cdot \frac{(2n+1)!}{x^{2n+1}} \right| \\ &= x^2 \cdot \lim_{n \rightarrow \infty} \frac{1}{(2n+3)(2n+2)} = 0 < 1 \end{aligned}$$

for all  $x$ , so  $R = \infty$ .

10.

$n$	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\cosh x$	1
1	$\sinh x$	0
2	$\cosh x$	1
3	$\sinh x$	0
$\vdots$	$\vdots$	$\vdots$

$$f^{(n)}(0) = \begin{cases} 1 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases} \text{ so } \cosh x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$$

Use the Ratio Test to find  $R$ . If  $a_n = \frac{x^{2n}}{(2n)!}$ , then

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{x^{2n+2}}{(2n+2)!} \cdot \frac{(2n)!}{x^{2n}} \right| \\ &= x^2 \cdot \lim_{n \rightarrow \infty} \frac{1}{(2n+2)(2n+1)} = 0 < 1 \end{aligned}$$

for all  $x$ , so  $R = \infty$ .

11.

$n$	$f^{(n)}(x)$	$f^{(n)}(2)$
0	$1+x+x^2$	7
1	$1+2x$	5
2	2	2
3	0	0
4	0	0
$\vdots$	$\vdots$	$\vdots$

$$\begin{aligned} f(x) &= 7 + 5(x-2) + \frac{2}{2!}(x-2)^2 + \sum_{n=3}^{\infty} \frac{0}{n!}(x-2)^n \\ &= 7 + 5(x-2) + (x-2)^2 \end{aligned}$$

Since  $a_n = 0$  for large  $n$ ,  $R = \infty$ .