36. $s_{4n-1} = c_0 + c_1x + c_2x^2 + c_3x^3 + c_0x^4 + c_1x^5 + c_2x^6 + c_3x^7 + \cdots + c_3x^{4n-1}$

$= (c_0 + c_1x + c_2x^2 + c_3x^3)(1 + x^4 + x^8 + \cdots + x^{4n-4}) \to \dfrac{c_0 + c_1x + c_2x^2 + c_3x^3}{1 - x^4}$ as $n \to \infty$

[by (12.2.4) with $r = x^4$] for $|x^4| < 1 \Leftrightarrow |x| < 1$. Also s_{4n}, s_{4n+1}, s_{4n+2} have the same limits (for example, $s_{4n} = s_{4n-1} + c_0x^{4n}$ and $x^{4n} \to 0$ for $|x| < 1$). So if at least one of c_0, c_1, c_2, and c_3 is nonzero, then the interval of convergence is $(-1, 1)$ and $f(x) = \dfrac{c_0 + c_1x + c_2x^2 + c_3x^3}{1 - x^4}$.

37. We use the Root Test on the series $\sum c_nx^n$. We need $\lim\limits_{n\to\infty} \sqrt[n]{|c_nx^n|} = |x| \lim\limits_{n\to\infty} \sqrt[n]{|c_n|} = c\,|x| < 1$ for convergence, or $|x| < 1/c$, so $R = 1/c$.

38. Suppose $c_n \neq 0$. Applying the Ratio Test to the series $\sum c_n(x-a)^n$, we find that

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{c_{n+1}(x-a)^{n+1}}{c_n(x-a)^n}\right| = \lim_{n\to\infty}\frac{|x-a|}{|c_n/c_{n+1}|}\,(*) = \frac{|x-a|}{\lim\limits_{n\to\infty}|c_n/c_{n+1}|}$$ (if $\lim\limits_{n\to\infty}|c_n/c_{n+1}| \neq 0$), so the series converges when $\dfrac{|x-a|}{\lim\limits_{n\to\infty}|c_n/c_{n+1}|} < 1 \Leftrightarrow |x-a| < \lim\limits_{n\to\infty}\left|\dfrac{c_n}{c_{n+1}}\right|$. Thus, $R = \lim\limits_{n\to\infty}\left|\dfrac{c_n}{c_{n+1}}\right|$. If $\lim\limits_{n\to\infty}\left|\dfrac{c_n}{c_{n+1}}\right| = 0$ and $|x-a| \neq 0$, then $(*)$ shows that $L = \infty$ and so the series diverges, and hence, $R = 0$. Thus, in all cases, $R = \lim\limits_{n\to\infty}\left|\dfrac{c_n}{c_{n+1}}\right|$.

39. For $2 < x < 3$, $\sum c_nx^n$ diverges and $\sum d_nx^n$ converges. By Exercise 12.2.61, $\sum(c_n + d_n)\,x^n$ diverges. Since both series converge for $|x| < 2$, the radius of convergence of $\sum(c_n + d_n)\,x^n$ is 2.

40. Since $\sum c_nx^n$ converges whenever $|x| < R$, $\sum c_nx^{2n} = \sum c_n(x^2)^n$ converges whenever $|x^2| < R \Leftrightarrow |x| < \sqrt{R}$, so the second series has radius of convergence $\sqrt{R}$.

12.9 Representations of Functions as Power Series

1. If $f(x) = \sum\limits_{n=0}^{\infty} c_nx^n$ has radius of convergence 10, then $f'(x) = \sum\limits_{n=1}^{\infty} nc_nx^{n-1}$ also has radius of convergence 10 by Theorem 2.

2. If $f(x) = \sum\limits_{n=0}^{\infty} b_nx^n$ converges on $(-2, 2)$, then $\int f(x)dx = C + \sum\limits_{n=0}^{\infty} \dfrac{b_n}{n+1}x^{n+1}$ has the same radius of convergence (by Theorem 2), but may not have the same interval of convergence—it may happen that the integrated series converges at an endpoint (or both endpoints).

3. Our goal is to write the function in the form $\dfrac{1}{1-r}$, and then use Equation (1) to represent the function as a sum of a power series. $f(x) = \dfrac{1}{1+x} = \dfrac{1}{1-(-x)} = \sum\limits_{n=0}^{\infty}(-x)^n = \sum\limits_{n=0}^{\infty}(-1)^nx^n$ with $|-x| < 1 \Leftrightarrow |x| < 1$, so $R = 1$ and $I = (-1, 1)$.

4. $f(x) = \dfrac{3}{1 - x^4} = 3\left(\dfrac{1}{1 - x^4}\right) = 3(1 + x^4 + x^8 + x^{12} + \cdots) = 3\sum\limits_{n=0}^{\infty}(x^4)^n = \sum\limits_{n=0}^{\infty} 3x^{4n}$ with $|x^4| < 1 \iff |x| < 1$, so $R = 1$ and $I = (-1, 1)$. [Note that $3\sum\limits_{n=0}^{\infty}(x^4)^n$ converges $\iff \sum\limits_{n=0}^{\infty}(x^4)^n$ converges, so the appropriate condition (from equation (1)) is $|x^4| < 1$.]

5. Replacing x with x^3 in (1) gives $f(x) = \dfrac{1}{1 - x^3} = \sum\limits_{n=0}^{\infty}(x^3)^n = \sum\limits_{n=0}^{\infty} x^{3n}$. The series converges when $|x^3| < 1 \iff |x|^3 < 1 \iff |x| < \sqrt[3]{1} \iff |x| < 1$. Thus, $R = 1$ and $I = (-1, 1)$.

6. $f(x) = \dfrac{1}{1 + 9x^2} = \dfrac{1}{1 - (-9x^2)} = \sum\limits_{n=0}^{\infty}(-9x^2)^n = \sum\limits_{n=0}^{\infty}(-1)^n 3^{2n} x^{2n}$. The series converges when $|-9x^2| < 1$; that is, when $|x| < \frac{1}{3}$, so $I = \left(-\frac{1}{3}, \frac{1}{3}\right)$.

7. $f(x) = \dfrac{1}{x - 5} = -\dfrac{1}{5}\left(\dfrac{1}{1 - x/5}\right) = -\dfrac{1}{5}\sum\limits_{n=0}^{\infty}\left(\dfrac{x}{5}\right)^n$ or equivalently, $-\sum\limits_{n=0}^{\infty}\dfrac{1}{5^{n+1}}x^n$. The series converges when $\left|\dfrac{x}{5}\right| < 1$; that is, when $|x| < 5$, so $I = (-5, 5)$.

8. $f(x) = \dfrac{x}{4x + 1} = x \cdot \dfrac{1}{1 - (-4x)} = x\sum\limits_{n=0}^{\infty}(-4x)^n = \sum\limits_{n=0}^{\infty}(-1)^n 2^{2n} x^{n+1}$. The series converges when $|-4x| < 1$; that is, when $|x| < \frac{1}{4}$, so $I = \left(-\frac{1}{4}, \frac{1}{4}\right)$.

9. $f(x) = \dfrac{x}{9 + x^2} = \dfrac{x}{9}\left[\dfrac{1}{1 + (x/3)^2}\right] = \dfrac{x}{9}\left[\dfrac{1}{1 - \{-(x/3)^2\}}\right] = \dfrac{x}{9}\sum\limits_{n=0}^{\infty}\left[-\left(\dfrac{x}{3}\right)^2\right]^n$

$= \dfrac{x}{9}\sum\limits_{n=0}^{\infty}(-1)^n\dfrac{x^{2n}}{9^n} = \sum\limits_{n=0}^{\infty}(-1)^n\dfrac{x^{2n+1}}{9^{n+1}}$. The geometric series $\sum\limits_{n=0}^{\infty}\left[-\left(\dfrac{x}{3}\right)^2\right]^n$ converges when

$\left|-\left(\dfrac{x}{3}\right)^2\right| < 1 \iff \dfrac{|x^2|}{9} < 1 \iff |x|^2 < 9 \iff |x| < 3$, so $R = 3$ and $I = (-3, 3)$.

10. $f(x) = \dfrac{x^2}{a^3 - x^3} = \dfrac{x^2}{a^3} \cdot \dfrac{1}{1 - x^3/a^3} = \dfrac{x^2}{a^3}\sum\limits_{n=0}^{\infty}\left(\dfrac{x^3}{a^3}\right)^n = \sum\limits_{n=0}^{\infty}\dfrac{x^{3n+2}}{a^{3n+3}}$. The series converges when $|x^3/a^3| < 1 \iff |x^3| < |a^3| \iff |x| < |a|$, so $R = |a|$ and $I = (-|a|, |a|)$.

11. $f(x) = \dfrac{3}{x^2 + x - 2} = \dfrac{3}{(x + 2)(x - 1)} = \dfrac{A}{x + 2} + \dfrac{B}{x - 1} \Rightarrow 3 = A(x - 1) + B(x + 2)$. Taking $x = -2$, we get $A = -1$. Taking $x = 1$, we get $B = 1$. Thus,

$$\frac{3}{x^2 + x - 2} = \frac{1}{x - 1} - \frac{1}{x + 2} = -\frac{1}{1 - x} - \frac{1}{2}\frac{1}{1 + x/2} = -\sum_{n=0}^{\infty} x^n - \frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^n$$

$$= \sum_{n=0}^{\infty}\left[-1 - \tfrac{1}{2}\left(-\tfrac{1}{2}\right)^n\right]x^n = \sum_{n=0}^{\infty}\left[-1 + \left(-\tfrac{1}{2}\right)^{n+1}\right]x^n = \sum_{n=0}^{\infty}\left[\frac{(-1)^{n+1}}{2^{n+1}} - 1\right]x^n$$

We represented the given function as the sum of two geometric series; the first converges for $x \in (-1, 1)$ and the second converges for $x \in (-2, 2)$. Thus, the sum converges for $x \in (-1, 1) = I$.

$-1)^n \left(\frac{x}{5}\right)^{2n}.$

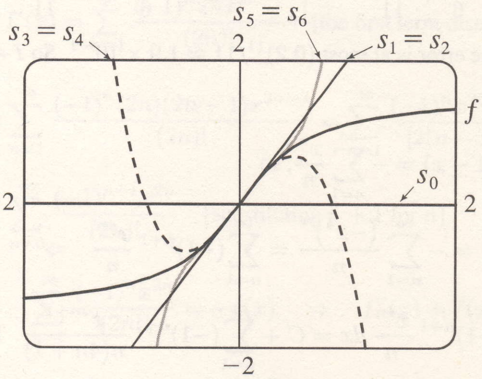
of the series are

22. $f(x) = \tan^{-1}(2x) = 2\int \frac{dx}{1+4x^2} = 2\int \sum_{n=0}^{\infty} (-1)^n \left(4x^2\right)^n dx = 2\int \sum_{n=0}^{\infty} (-1)^n 4^n x^{2n}\, dx$

$= C + 2\sum_{n=0}^{\infty} \frac{(-1)^n 4^n x^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1} x^{2n+1}}{2n+1} \quad [f(0) = \tan^{-1} 0 = 0, \text{ so } C = 0].$

The series converges when $\left|4x^2\right| < 1 \iff |x| < \frac{1}{2}$, so $R = \frac{1}{2}$. If $x = \pm\frac{1}{2}$, then $f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1}$ and

$f(x) = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{1}{2n+1}$, respectively. Both series converge by the Alternating Series Test.



As n increases, $s_n(x)$ approximates f better on the interval of convergence, which is $\left[-\frac{1}{2}, \frac{1}{2}\right]$.

23. $\frac{t}{1-t^8} = t \cdot \frac{1}{1-t^8} = t\sum_{n=0}^{\infty} (t^8)^n = \sum_{n=0}^{\infty} t^{8n+1} \Rightarrow \int \frac{t}{1-t^8}\, dt = C + \sum_{n=0}^{\infty} \frac{t^{8n+2}}{8n+2}$. The series for $\frac{1}{1-t^8}$ converges when $\left|t^8\right| < 1 \iff |t| < 1$, so $R = 1$ for that series and also the series for $t/(1-t^8)$. By Theorem 2, the series for $\int \frac{t}{1-t^8}\, dt$ also has $R = 1$.

24. By Example 6, $\ln(1-t) = -\sum_{n=1}^{\infty} \frac{t^n}{n}$ for $|t| < 1$, so $\frac{\ln(1-t)}{t} = -\sum_{n=1}^{\infty} \frac{t^{n-1}}{n}$ and

$\int \frac{\ln(1-t)}{t}\, dt = C - \sum_{n=1}^{\infty} \frac{t^n}{n^2}$. By Theorem 2, $R = 1$.

25. By Example 7, $\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$ with $R = 1$, so

$x - \tan^{-1} x = x - \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots\right) = \frac{x^3}{3} - \frac{x^5}{5} + \frac{x^7}{7} - \cdots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n+1}}{2n+1}$

and $\frac{x - \tan^{-1} x}{x^3} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n-2}}{2n+1}$, so

$\int \frac{x - \tan^{-1} x}{x^3}\, dx = C + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n-1}}{(2n+1)(2n-1)} = C + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n-1}}{4n^2-1}$. By Theorem 2, $R = 1$.