4. If $a_n = \dfrac{(-1)^n x^n}{n+1}$, then $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{x^{n+1}}{n+2}\cdot\dfrac{n+1}{x^n}\right| = \lim\limits_{n\to\infty}\dfrac{|x|}{1+1/(n+1)} = |x|$. By the Ratio Test, the series $\sum\limits_{n=0}^{\infty}\dfrac{(-1)^n x^n}{n+1}$ converges when $|x|<1$, so $R=1$. When $x=-1$, the series diverges because it is the harmonic series; when $x=1$, it is the alternating harmonic series, which converges by the Alternating Series Test. Thus, $I=(-1,1]$.

5. If $a_n = \dfrac{(-1)^{n-1}x^n}{n^3}$, then $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{(-1)^n x^{n+1}}{(n+1)^3}\cdot\dfrac{n^3}{(-1)^{n-1}x^n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{(-1)xn^3}{(n+1)^3}\right|$ $= \lim\limits_{n\to\infty}\left[\left(\dfrac{n}{n+1}\right)^3|x|\right] = 1^3\cdot|x| = |x|$. By the Ratio Test, the series $\sum\limits_{n=1}^{\infty}\dfrac{(-1)^{n-1}x^n}{n^3}$ converges when $|x|<1$, so the radius of convergence $R=1$. Now we'll check the endpoints, that is, $x=\pm1$. When $x=1$, the series $\sum\limits_{n=1}^{\infty}\dfrac{(-1)^{n-1}}{n^3}$ converges by the Alternating Series Test. When $x=-1$, the series $\sum\limits_{n=1}^{\infty}\dfrac{(-1)^{n-1}(-1)^n}{n^3} = -\sum\limits_{n=1}^{\infty}\dfrac{1}{n^3}$ converges because it is a constant multiple of a convergent p-series ($p=3>1$). Thus, the interval of convergence is $I=[-1,1]$.

6. $a_n=\sqrt{n}\,x^n$, so we need $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\dfrac{\sqrt{n+1}\,|x|^{n+1}}{\sqrt{n}\,|x|^n} = \lim\limits_{n\to\infty}\sqrt{1+\dfrac{1}{n}}\,|x| = |x| < 1$ for convergence (by the Ratio Test), so $R=1$. When $x=\pm1$, $\lim\limits_{n\to\infty}|a_n| = \lim\limits_{n\to\infty}\sqrt{n}=\infty$, so the series diverges by the Test for Divergence. Thus, $I=(-1,1)$.

7. If $a_n=\dfrac{x^n}{n!}$, then $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{x^{n+1}}{(n+1)!}\cdot\dfrac{n!}{x^n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{x}{n+1}\right| = |x|\lim\limits_{n\to\infty}\dfrac{1}{n+1} = |x|\cdot 0 = 0 < 1$ for *all* real x. So, by the Ratio Test, $R=\infty$, and $I=(-\infty,\infty)$.

8. Here the Root Test is easier. If $a_n=n^n x^n$ then $\lim\limits_{n\to\infty}\sqrt[n]{|a_n|} = \lim\limits_{n\to\infty} n\,|x| = \infty$ if $x\neq 0$, so $R=0$ and $I=\{0\}$.

9. $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\dfrac{(n+1)\,4^{n+1}\,|x|^{n+1}}{n4^n\,|x|^n} = \lim\limits_{n\to\infty}\left(1+\dfrac{1}{n}\right)4\,|x| = 4\,|x|$. Now $4\,|x|<1 \;\Leftrightarrow\; |x|<\frac14$, so by the Ratio Test, $R=\frac14$. When $x=\frac14$, we get the divergent series $\sum_{n=1}^{\infty}(-1)^n n$, and when $x=-\frac14$, we get the divergent series $\sum_{n=1}^{\infty} n$. Thus, $I=\left(-\frac14,\frac14\right)$.

10. If $a_n=\dfrac{x^n}{n3^n}$, then $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{x^{n+1}}{(n+1)3^{n+1}}\cdot\dfrac{n3^n}{x^n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{xn}{(n+1)3}\right| = \dfrac{|x|}{3}\lim\limits_{n\to\infty}\dfrac{n}{n+1} = \dfrac{|x|}{3}$. By the Ratio Test, the series converges when $\dfrac{|x|}{3}<1 \;\Leftrightarrow\; |x|<3$, so $R=3$. When $x=-3$, the series is the alternating harmonic series, which converges by the Alternating Series Test. When $x=3$, it is the harmonic series, which diverges. Thus, $I=[-3,3)$.

11. $a_n=\dfrac{(-2)^n x^n}{\sqrt[4]{n}}$, so $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\dfrac{2^{n+1}\,|x|^{n+1}}{\sqrt[4]{n+1}}\cdot\dfrac{\sqrt[4]{n}}{2^n\,|x|^n} = \lim\limits_{n\to\infty} 2\,|x|\sqrt[4]{\dfrac{n}{n+1}} = 2\,|x|$, so by the Ratio Test, the series converges when $2\,|x|<1 \;\Leftrightarrow\; |x|<\frac12$, so $R=\frac12$. When $x=-\frac12$, we get the divergent

p-series $\sum_{n=1}^{\infty} \frac{1}{\sqrt[4]{n}}$ $\left(p = \frac{1}{4} \leq 1\right)$. When $x = \frac{1}{2}$, we get the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[4]{n}}$, which converges by the Alternating Series Test. Thus, $I = \left(-\frac{1}{2}, \frac{1}{2}\right]$.

12. $a_n = \dfrac{x^n}{5^n n^5}$, so $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \left|\dfrac{x^{n+1}}{5^{n+1}(n+1)^5} \cdot \dfrac{5^n n^5}{x^n}\right| = \lim\limits_{n\to\infty} \dfrac{|x|}{5}\left(\dfrac{n}{n+1}\right)^5 = \dfrac{|x|}{5}$. By the Ratio Test, the series converges when $|x|/5 < 1 \;\Leftrightarrow\; |x| < 5$, so $R = 5$. When $x = -5$, we get the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^5}$, which converges by the Alternating Series Test. When $x = 5$, we get the convergent p-series $\sum_{n=1}^{\infty} \frac{1}{n^5}$ $(p = 5 > 1)$. Thus, $I = [-5, 5]$.

13. If $a_n = (-1)^n \dfrac{x^n}{4^n \ln n}$, then

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}}{4^{n+1} \ln(n+1)} \cdot \frac{4^n \ln n}{x^n}\right| = \frac{|x|}{4} \lim_{n\to\infty} \frac{\ln n}{\ln(n+1)} = \frac{|x|}{4} \cdot 1 \text{ (by l'Hospital's Rule)} = \frac{|x|}{4}.$$

By the Ratio Test, the series converges when $\dfrac{|x|}{4} < 1 \;\Leftrightarrow\; |x| < 4$, so $R = 4$. When $x = -4$, $\sum_{n=2}^{\infty} (-1)^n \dfrac{x^n}{4^n \ln n} = \sum_{n=2}^{\infty} \dfrac{[(-1)(-4)]^n}{4^n \ln n} = \sum_{n=2}^{\infty} \dfrac{1}{\ln n}$. Since $\ln n < n$ for $n \geq 2$, $\dfrac{1}{\ln n} > \dfrac{1}{n}$ and $\sum_{n=2}^{\infty} \dfrac{1}{n}$ is the divergent harmonic series (without the $n = 1$ term), $\sum_{n=2}^{\infty} \dfrac{1}{\ln n}$ is divergent by the Comparison Test. When $x = 4$, $\sum_{n=2}^{\infty} (-1)^n \dfrac{x^n}{4^n \ln n} = \sum_{n=2}^{\infty} (-1)^n \dfrac{1}{\ln n}$, which converges by the Alternating Series Test. Thus, $I = (-4, 4]$.

14. $a_n = (-1)^n \dfrac{x^{2n}}{(2n)!}$, so $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \dfrac{|x|^{2n+2}}{(2n+2)!} \cdot \dfrac{(2n)!}{|x|^{2n}} = \lim\limits_{n\to\infty} \dfrac{|x|^2}{(2n+1)(2n+2)} = 0$. Thus, by the Ratio Test, the series converges for *all* real x and we have $R = \infty$ and $I = (-\infty, \infty)$.

15. If $a_n = \sqrt{n}\,(x-1)^n$, then $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \left|\dfrac{\sqrt{n+1}\,|x-1|^{n+1}}{\sqrt{n}\,|x-1|^n}\right| = \lim\limits_{n\to\infty} \sqrt{1 + \dfrac{1}{n}}\,|x-1| = |x-1|$. By the Ratio Test, the series converges when $|x-1| < 1$ [so $R = 1$] $\;\Leftrightarrow\; -1 < x - 1 < 1 \;\Leftrightarrow\; 0 < x < 2$. When $x = 0$, the series becomes $\sum_{n=0}^{\infty} (-1)^n \sqrt{n}$, which diverges by the Test for Divergence. When $x = 2$, the series becomes $\sum_{n=0}^{\infty} \sqrt{n}$, which also diverges by the Test for Divergence. Thus, $I = (0, 2)$.

16. If $a_n = n^3 (x-5)^n$, $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \left|\dfrac{(n+1)^3 (x-5)^{n+1}}{n^3 (x-5)^n}\right| = \lim\limits_{n\to\infty} \left(1 + \dfrac{1}{n}\right)^3 |x-5| = |x-5|$. By the Ratio Test, the series converges when $|x-5| < 1 \;\Leftrightarrow\; -1 < x - 5 < 1 \;\Leftrightarrow\; 4 < x < 6$. When $x = 4$, the series becomes $\sum_{n=0}^{\infty} (-1)^n n^3$, which diverges by the Test for Divergence. When $x = 6$, the series becomes $\sum_{n=0}^{\infty} n^3$, which also diverges by the Test for Divergence. Thus, $R = 1$ and $I = (4, 6)$.

17. If $a_n = (-1)^n \dfrac{(x+2)^n}{n2^n}$, then

$\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left[\frac{|x+2|^{n+1}}{(n+1)2^{n+1}}\cdot\frac{n2^n}{|x+2|^n}\right] = \lim_{n\to\infty}\frac{n}{n+1}\cdot\frac{|x+2|}{2} = \frac{|x+2|}{2}$. By the Ratio Test, the series converges when $\dfrac{|x+2|}{2} < 1 \;\Leftrightarrow\; |x+2| < 2$ [so $R = 2$] $\Leftrightarrow\; -2 < x+2 < 2 \;\Leftrightarrow\; -4 < x < 0$.

When $x = -4$, the series becomes $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{(-2)^n}{n2^n} = \sum_{n=1}^{\infty}\frac{2^n}{n2^n} = \sum_{n=1}^{\infty}\frac{1}{n}$, which is the divergent harmonic series.

When $x = 0$, the series is $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$, the alternating harmonic series, which converges by the Alternating Series Test. Thus, $I = (-4, 0]$.

18. If $a_n = \dfrac{(-2)^n}{\sqrt{n}}(x+3)^n$, then

$\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-2)^{n+1}(x+3)^{n+1}}{\sqrt{n+1}}\cdot\frac{\sqrt{n}}{(-2)^n(x+3)^n}\right| = \lim_{n\to\infty}\frac{2\,|x+3|}{\sqrt{1+1/n}} = 2\,|x+3| < 1 \;\Leftrightarrow$

$|x+3| < \frac{1}{2}$ [so $R = \frac{1}{2}$] $\;\Leftrightarrow\; -\frac{7}{2} < x < -\frac{5}{2}$. When $x = -\frac{7}{2}$, the series becomes $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$, which diverges because it is a p-series with $p = \frac{1}{2} \le 1$. When $x = -\frac{5}{2}$, the series becomes $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$, which converges by the Alternating Series Test. Thus, $I = \left(-\frac{7}{2}, -\frac{5}{2}\right]$.

19. If $a_n = \dfrac{(x-2)^n}{n^n}$, then $\displaystyle\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\frac{|x-2|}{n} = 0$, so the series converges for all x (by the Root Test). $R = \infty$ and $I = (-\infty, \infty)$.

20. $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(3x-2)^{n+1}}{(n+1)\,3^{n+1}}\cdot\frac{n3^n}{(3x-2)^n}\right| = \lim_{n\to\infty}\left(\frac{|3x-2|}{3}\cdot\frac{1}{1+1/n}\right) = \frac{|3x-2|}{3} = \left|x-\frac{2}{3}\right|$, so by the Ratio Test, the series converges when $\left|x-\frac{2}{3}\right| < 1 \;\Leftrightarrow\; -\frac{1}{3} < x < \frac{5}{3}$. $R = 1$. When $x = -\frac{1}{3}$, the series is $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$, the convergent alternating harmonic series. When $x = \frac{5}{3}$, the series becomes the divergent harmonic series. Thus, $I = \left[-\frac{1}{3}, \frac{5}{3}\right)$.

21. $a_n = \dfrac{n}{b^n}(x-a)^n$, where $b > 0$.

$\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+1)\,|x-a|^{n+1}}{b^{n+1}}\cdot\frac{b^n}{n\,|x-a|^n} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)\frac{|x-a|}{b} = \frac{|x-a|}{b}$.

By the Ratio Test, the series converges when $\dfrac{|x-a|}{b} < 1 \;\Leftrightarrow\; |x-a| < b$ [so $R = b$] $\;\Leftrightarrow$ $-b < x-a < b \;\Leftrightarrow\; a-b < x < a+b$. When $|x-a| = b$, $\displaystyle\lim_{n\to\infty}|a_n| = \lim_{n\to\infty} n = \infty$, so the series diverges. Thus, $I = (a-b, a+b)$.

22. $a_n = \dfrac{n(x-4)^n}{n^3+1}$, so

$\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+1)\,|x-4|^{n+1}}{(n+1)^3+1}\cdot\frac{n^3+1}{n\,|x-4|^n} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)\frac{n^3+1}{n^3+3n^2+3n+2}\,|x-4| = |x-4|$.