12.5 Alternating Series

1. (a) An alternating series is a series whose terms are alternately positive and negative.

(b) An alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}b_n$ converges if $0 < b_{n+1} \le b_n$ for all n and $\lim\limits_{n\to\infty} b_n = 0$. (This is the Alternating Series Test.)

(c) The error involved in using the partial sum s_n as an approximation to the total sum s is the remainder $R_n = s - s_n$ and the size of the error is smaller than b_{n+1}; that is, $|R_n| \le b_{n+1}$. (This is the Alternating Series Estimation Theorem.)

2. $-\frac{1}{3}+\frac{2}{4}-\frac{3}{5}+\frac{4}{6}-\frac{5}{7}+\cdots=\sum\limits_{n=1}^{\infty}(-1)^n\frac{n}{n+2}$. Here $a_n = (-1)^n\frac{n}{n+2}$. Since $\lim\limits_{n\to\infty} a_n \ne 0$ (in fact the limit does not exist), the series diverges by the Test for Divergence.

3. $\frac{4}{7}-\frac{4}{8}+\frac{4}{9}-\frac{4}{10}+\frac{4}{11}-\cdots=\sum\limits_{n=1}^{\infty}(-1)^{n-1}\frac{4}{n+6}$. Now $b_n = \frac{4}{n+6} > 0$, $\{b_n\}$ is decreasing, and $\lim\limits_{n\to\infty} b_n = 0$, so the series converges by the Alternating Series Test.

4. $\sum\limits_{n=2}^{\infty}(-1)^n\frac{1}{\ln n}$. $b_n = \frac{1}{\ln n}$ is positive and $\{b_n\}$ is decreasing; $\lim\limits_{n\to\infty}\frac{1}{\ln n} = 0$, so the series converges by the Alternating Series Test.

5. $b_n = \frac{1}{\sqrt{n}} > 0$, $\{b_n\}$ is decreasing, and $\lim\limits_{n\to\infty} b_n = 0$, so the series $\sum\limits_{n=1}^{\infty}\frac{(-1)^{n-1}}{\sqrt{n}}$ converges by the Alternating Series Test.

6. $b_n = \frac{1}{3n-1} > 0$, $\{b_n\}$ is decreasing, and $\lim\limits_{n\to\infty} b_n = 0$, so the series $\sum\limits_{n=1}^{\infty}\frac{(-1)^{n-1}}{3n-1}$ converges by the Alternating Series Test.

7. $\sum\limits_{n=1}^{\infty} a_n = \sum\limits_{n=1}^{\infty}(-1)^n\frac{3n-1}{2n+1} = \sum\limits_{n=1}^{\infty}(-1)^n b_n$. Now $\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty}\frac{3-1/n}{2+1/n} = \frac{3}{2} \ne 0$. Since $\lim\limits_{n\to\infty} a_n \ne 0$ (in fact the limit does not exist), the series diverges by the Test for Divergence.

8. $b_n = \frac{2n}{4n^2+1} > 0$, $\{b_n\}$ is decreasing [since

$$b_n - b_{n+1} = \frac{2n}{4n^2+1} - \frac{2n+2}{4n^2+8n+5} = \frac{8n^2+8n-2}{(4n^2+1)(4n^2+8n+5)} > 0 \text{ for } n \ge 1\text{], and}$$

$\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty}\frac{2/n}{4+1/n^2} = 0$, so the series $\sum\limits_{n=1}^{\infty}(-1)^n\frac{2n}{4n^2+1}$ converges by the Alternating Series Test.

Alternatively, to show that $\{b_n\}$ is decreasing, we could verify that $\frac{d}{dx}\left(\frac{2x}{4x^2+1}\right) < 0$ for $x \ge 1$.

9. $b_n = \frac{1}{4n^2+1} > 0$, $\{b_n\}$ is decreasing, and $\lim\limits_{n\to\infty} b_n = 0$, so the series $\sum\limits_{n=1}^{\infty}\frac{(-1)^{n+1}}{4n^2+1}$ converges by the Alternating Series Test.

10. $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}}{1+2\sqrt{n}} = \sum_{n=1}^{\infty} (-1)^n b_n$. Now $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{2+1/\sqrt{n}} = \frac{1}{2} \neq 0$. Since $\lim_{n\to\infty} a_n \neq 0$ (in fact the limit does not exist), the series diverges by the Test for Divergence.

11. $b_n = \frac{n^2}{n^3+4} > 0$ for $n \geq 1$. $\{b_n\}$ is decreasing for $n \geq 2$ since

$$\left(\frac{x^2}{x^3+4}\right)' = \frac{(x^3+4)(2x) - x^2(3x^2)}{(x^3+4)^2} = \frac{x(2x^3+8-3x^3)}{(x^3+4)^2} = \frac{x(8-x^3)}{(x^3+4)^2} < 0 \text{ for } x > 2. \text{ Also,}$$

$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1/n}{1+4/n^3} = 0$. Thus, the series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3+4}$ converges by the Alternating Series Test.

12. $b_n = \frac{e^{1/n}}{n} > 0$ for $n \geq 1$. $\{b_n\}$ is decreasing since

$$\left(\frac{e^{1/x}}{x}\right)' = \frac{x \cdot e^{1/x}(-1/x^2) - e^{1/x} \cdot 1}{x^2} = \frac{-e^{1/x}(1+x)}{x^3} < 0 \text{ for } x > 0. \text{ Also, } \lim_{n\to\infty} b_n = 0 \text{ since}$$

$\lim_{n\to\infty} e^{1/n} = 1$. Thus, the series $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{e^{1/n}}{n}$ converges by the Alternating Series Test.

13. $\sum_{n=2}^{\infty} (-1)^n \frac{n}{\ln n}$. $\lim_{n\to\infty} \frac{n}{\ln n} = \lim_{x\to\infty} \frac{x}{\ln x} \overset{H}{=} \lim_{x\to\infty} \frac{1}{1/x} = \infty$, so the series diverges by the Test for Divergence.

14. $\sum_{n=1}^{\infty} (-1)^{n-1} \left(\frac{\ln n}{n}\right) = 0 + \sum_{n=2}^{\infty} (-1)^{n-1} \left(\frac{\ln n}{n}\right)$. $b_n = \frac{\ln n}{n} > 0$ for $n \geq 2$, and if $f(x) = \frac{\ln x}{x}$, then $f'(x) = \frac{1-\ln x}{x^2} < 0$ for $x > e$, so $\{b_n\}$ is eventually decreasing. Also,

$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{\ln n}{n} = \lim_{x\to\infty} \frac{\ln x}{x} \overset{H}{=} \lim_{x\to\infty} \frac{1/x}{1} = 0$, so the series converges by the Alternating Series Test.

15. $\sum_{n=1}^{\infty} \frac{\cos n\pi}{n^{3/4}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{3/4}}$. $b_n = \frac{1}{n^{3/4}}$ is decreasing and positive and $\lim_{n\to\infty} \frac{1}{n^{3/4}} = 0$, so the series converges by the Alternating Series Test.

16. $\sin\left(\frac{n\pi}{2}\right) = 0$ if n is even and $(-1)^k$ if $n = 2k+1$, so the series is $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}$. $b_n = \frac{1}{(2n+1)!} > 0$, $\{b_n\}$ is decreasing, and $\lim_{n\to\infty} \frac{1}{(2n+1)!} = 0$, so the series converges by the Alternating Series Test.

17. $\sum_{n=1}^{\infty} (-1)^n \sin \frac{\pi}{n}$. $b_n = \sin \frac{\pi}{n} > 0$ for $n \geq 2$ and $\sin \frac{\pi}{n} \geq \sin \frac{\pi}{n+1}$, and $\lim_{n\to\infty} \sin \frac{\pi}{n} = \sin 0 = 0$, so the series converges by the Alternating Series Test.

18. $\sum_{n=1}^{\infty} (-1)^n \cos\left(\frac{\pi}{n}\right)$. $\lim_{n\to\infty} \cos\left(\frac{\pi}{n}\right) = \cos(0) = 1$, so $\lim_{n\to\infty} (-1)^n \cos\left(\frac{\pi}{n}\right)$ does not exist and the series diverges by the Test for Divergence.

19. $\frac{n^n}{n!} = \frac{n \cdot n \cdot \cdots \cdot n}{1 \cdot 2 \cdot \cdots \cdot n} \geq n \Rightarrow \lim_{n\to\infty} \frac{n^n}{n!} = \infty \Rightarrow \lim_{n\to\infty} \frac{(-1)^n n^n}{n!}$ does not exist. So the series diverges by the Test for Divergence.