8. The function $f(x) = \dfrac{x+2}{x+1} = 1 + \dfrac{1}{x+1}$ is continuous, positive, and decreasing on $[1, \infty)$, so the Integral Test applies.

$$\int_1^\infty f(x)\,dx = \lim_{t\to\infty} \int_1^t \left(1 + \frac{1}{x+1}\right) dx = \lim_{t\to\infty} [x + \ln(x+1)]_1^t = \lim_{t\to\infty} (t + \ln(t+1) - 1 - \ln 2) = \infty, \text{ so}$$

$\displaystyle\int_1^\infty \frac{x+2}{x+1}\,dx$ is divergent and the series $\displaystyle\sum_{n=1}^\infty \frac{n+2}{n+1}$ is divergent. NOTE: $\displaystyle\lim_{n\to\infty} \frac{n+2}{n+1} = 1$, so the given series diverges by the Test for Divergence.

9. The series $\displaystyle\sum_{n=1}^\infty \frac{1}{n^{0.85}}$ is a p-series with $p = 0.85 \le 1$, so it diverges by (1). Therefore, the series $\displaystyle\sum_{n=1}^\infty \frac{2}{n^{0.85}}$ must also diverge, for if it converged, then $\displaystyle\sum_{n=1}^\infty \frac{1}{n^{0.85}}$ would have to converge (by Theorem 8(i) in Section 11.2).

10. $\displaystyle\sum_{n=1}^\infty n^{-1.4}$ and $\displaystyle\sum_{n=1}^\infty n^{-1.2}$ are p-series with $p > 1$, so they converge by (1). Thus, $\displaystyle\sum_{n=1}^\infty 3n^{-1.2}$ converges by Theorem 8(i) in Section 11.2. It follows from Theorem 8(ii) that the given series $\displaystyle\sum_{n=1}^\infty (n^{-1.4} + 3n^{-1.2})$ also converges.

11. $1 + \dfrac{1}{8} + \dfrac{1}{27} + \dfrac{1}{64} + \dfrac{1}{125} + \cdots = \displaystyle\sum_{n=1}^\infty \frac{1}{n^3}$. This is a p-series with $p = 3 > 1$, so it converges by (1).

12. $1 + \dfrac{1}{2\sqrt{2}} + \dfrac{1}{3\sqrt{3}} + \dfrac{1}{4\sqrt{4}} + \dfrac{1}{5\sqrt{5}} + \cdots = \displaystyle\sum_{n=1}^\infty \frac{1}{n\sqrt{n}} = \sum_{n=1}^\infty \frac{1}{n^{3/2}}$. This is a p-series with $p = \frac{3}{2} > 1$, so it converges by (1).

13. $\displaystyle\sum_{n=1}^\infty \frac{5 - 2\sqrt{n}}{n^3} = 5\sum_{n=1}^\infty \frac{1}{n^3} - 2\sum_{n=1}^\infty \frac{1}{n^{5/2}}$ by Theorem 12.2.8, since $\displaystyle\sum_{n=1}^\infty \frac{1}{n^3}$ and $\displaystyle\sum_{n=1}^\infty \frac{1}{n^{5/2}}$ both converge by (1) (with $p = 3 > 1$ and $p = \frac{5}{2} > 1$). Thus, $\displaystyle\sum_{n=1}^\infty \frac{5 - 2\sqrt{n}}{n^3}$ converges.

14. The function $f(x) = \dfrac{5}{x-2}$ is continuous, positive, and decreasing on $[3, \infty)$, so we can apply the Integral Test.

$\displaystyle\int_3^\infty \frac{5}{x-2}\,dx = \lim_{t\to\infty} \int_3^t \frac{5}{x-2}\,dx = \lim_{t\to\infty} [5\ln(x-2)]_3^t = \lim_{t\to\infty} [5\ln(t-2) - 0] = \infty$, so the series $\displaystyle\sum_{n=3}^\infty \frac{5}{n-2}$ diverges.

15. The function $f(x) = \dfrac{1}{x^2+4}$ is continuous, positive, and decreasing on $[1, \infty)$, so we can apply the Integral Test.

$$\int_1^\infty \frac{1}{x^2+4}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^2+4}\,dx = \lim_{t\to\infty} \left[\frac{1}{2}\tan^{-1}\frac{x}{2}\right]_1^t = \frac{1}{2}\lim_{t\to\infty} \left[\tan^{-1}\left(\frac{t}{2}\right) - \tan^{-1}\left(\frac{1}{2}\right)\right]$$

$$= \frac{1}{2}\left[\frac{\pi}{2} - \tan^{-1}\left(\frac{1}{2}\right)\right]$$

Therefore, the series $\displaystyle\sum_{n=1}^\infty \frac{1}{n^2+4}$ converges.

16. The function $f(x) = \dfrac{3x+2}{x(x+1)} = \dfrac{2}{x} + \dfrac{1}{x+1}$ [by partial fractions] is continuous, positive, and decreasing on $[1, \infty)$ since it is the sum of two such functions. Thus, we can apply the Integral Test.

$$\begin{aligned}\int_1^\infty \frac{3x+2}{x(x+1)}\,dx &= \lim_{t\to\infty}\int_1^t \left[\frac{2}{x} + \frac{1}{x+1}\right] dx = \lim_{t\to\infty}\left[2\ln x + \ln(x+1)\right]_1^t \\ &= \lim_{t\to\infty}\left[2\ln t + \ln(t+1) - \ln 2\right] = \infty\end{aligned}$$

Thus, the series $\displaystyle\sum_{n=1}^{\infty} \frac{3n+2}{n(n+1)}$ diverges.

17. $f(x) = \dfrac{x}{x^2+1}$ is continuous and positive on $[1, \infty)$, and since $f'(x) = \dfrac{1-x^2}{(x^2+1)^2} < 0$ for $x > 1$, f is also decreasing. Using the Integral Test,

$\displaystyle\int_1^\infty \frac{x}{x^2+1}\,dx = \lim_{t\to\infty}\int_1^t \frac{x}{x^2+1}\,dx = \lim_{t\to\infty}\left[\frac{\ln(x^2+1)}{2}\right]_1^t = \frac{1}{2}\lim_{t\to\infty}\left[\ln(t^2+1) - \ln 2\right] = \infty$, so the series diverges.

18. The function $f(x) = \dfrac{1}{x^2-4x+5} = \dfrac{1}{(x-2)^2+1}$ is continuous, positive, and decreasing on $[2, \infty)$, so the Integral Test applies. $\displaystyle\int_2^\infty f(x)\,dx = \lim_{t\to\infty}\int_2^t f(x)\,dx = \lim_{t\to\infty}\int_2^t \frac{1}{(x-2)^2+1}\,dx = \lim_{t\to\infty}\left[\tan^{-1}(x-2)\right]_2^t =$ $\displaystyle\lim_{t\to\infty}\left[\tan^{-1}(t-2) - \tan^{-1} 0\right] = \frac{\pi}{2} - 0 = \frac{\pi}{2}$, so the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2-4n+5}$ converges. Of course this means that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2-4n+5}$ converges too.

19. $f(x) = xe^{-x^2}$ is continuous and positive on $[1, \infty)$, and since $f'(x) = e^{-x^2}(1-2x^2) < 0$ for $x > 1$, f is decreasing as well. Thus, we can use the Integral Test.

$\int_1^\infty xe^{-x^2}\,dx = \lim\limits_{t\to\infty}\left[-\frac{1}{2}e^{-x^2}\right]_1^t = 0 - \left(-\frac{1}{2}e^{-1}\right) = 1/(2e)$. Since the integral converges, the series converges.

20. $f(x) = \dfrac{\ln x}{x^2}$ is continuous and positive for $x \geq 2$, and $f'(x) = \dfrac{1-2\ln x}{x^3} < 0$ for $x \geq 2$, so f is decreasing.

$\displaystyle\int_2^\infty \frac{\ln x}{x^2}\,dx = \lim_{t\to\infty}\left[-\frac{\ln x}{x} - \frac{1}{x}\right]_2^t$ [by parts] $\overset{\text{H}}{=} 1$. Thus, $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^2} = \sum_{n=2}^{\infty} \frac{\ln n}{n^2}$ converges by the Integral Test.

21. $f(x) = \dfrac{1}{x\ln x}$ is continuous and positive on $[2, \infty)$, and also decreasing since $f'(x) = -\dfrac{1+\ln x}{x^2(\ln x)^2} < 0$ for $x > 2$, so we can use the Integral Test. $\displaystyle\int_2^\infty \frac{1}{x\ln x}\,dx = \lim_{t\to\infty}\left[\ln(\ln x)\right]_2^t = \lim_{t\to\infty}\left[\ln(\ln t) - \ln(\ln 2)\right] = \infty$, so the series diverges.

22. The function $f(x) = \dfrac{x}{x^4+1}$ is positive, continuous, and decreasing on $[1, \infty)$. [Note that

$f'(x) = \dfrac{x^4+1-4x^4}{(x^4+1)^2} = \dfrac{1-3x^4}{(x^4+1)^2} < 0$ on $[1, \infty)$.] Thus, we can apply the Integral Test.

$$\int_1^\infty \frac{x}{x^4+1}\,dx = \lim_{t\to\infty}\int_1^t \frac{\frac{1}{2}(2x)}{1+(x^2)^2}\,dx = \lim_{t\to\infty}\left[\frac{1}{2}\tan^{-1}(x^2)\right]_1^t = \frac{1}{2}\lim_{t\to\infty}\left[\tan^{-1}(t^2) - \tan^{-1} 1\right]$$
$$= \frac{1}{2}\left(\frac{\pi}{2} - \frac{\pi}{4}\right) = \frac{\pi}{8}$$

so the series $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^4+1}$ converges.

23. The function $f(x) = \dfrac{1}{x^3+x}$ is continuous, positive, and decreasing on $[1, \infty)$, so the Integral Test applies. We use partial fractions to evaluate the integral:

$$\int_1^\infty \frac{1}{x^3+x}\,dx = \lim_{t\to\infty}\int_1^t \left[\frac{1}{x} - \frac{x}{1+x^2}\right]dx = \lim_{t\to\infty}\left[\ln x - \frac{1}{2}\ln(1+x^2)\right]_1^t$$
$$= \lim_{t\to\infty}\left[\ln\frac{x}{\sqrt{1+x^2}}\right]_1^t = \lim_{t\to\infty}\left(\ln\frac{t}{\sqrt{1+t^2}} - \ln\frac{1}{\sqrt{2}}\right)$$
$$= \lim_{t\to\infty}\left(\ln\frac{1}{\sqrt{1+1/t^2}} + \frac{1}{2}\ln 2\right) = \frac{1}{2}\ln 2$$

so the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3+n}$ converges.

24. $f(x) = \dfrac{1}{x\ln x\ln(\ln x)}$ is positive and continuous on $[3, \infty)$, and is decreasing since x, $\ln x$, and $\ln(\ln x)$ are all increasing; so we can apply the Integral Test. $\displaystyle\int_3^\infty \frac{dx}{x\ln x\ln(\ln x)} = \lim_{t\to\infty}\left[\ln(\ln(\ln x))\right]_3^t = \infty$. The integral diverges, so $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n\ln n\ln(\ln n)}$ diverges.

25. We have already shown (in Exercise 21) that when $p = 1$ the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ diverges, so assume that $p \neq 1$.

$f(x) = \dfrac{1}{x(\ln x)^p}$ is continuous and positive on $[2, \infty)$, and $f'(x) = -\dfrac{p+\ln x}{x^2(\ln x)^{p+1}} < 0$ if $x > e^{-p}$, so that f is eventually decreasing and we can use the Integral Test.

$$\int_2^\infty \frac{1}{x(\ln x)^p}\,dx = \lim_{t\to\infty}\left[\frac{(\ln x)^{1-p}}{1-p}\right]_2^t \quad (\text{for } p \neq 1) \quad = \lim_{t\to\infty}\left[\frac{(\ln t)^{1-p}}{1-p}\right] - \frac{(\ln 2)^{1-p}}{1-p}$$

This limit exists whenever $1 - p < 0 \Leftrightarrow p > 1$, so the series converges for $p > 1$.

26. As in Exercise 24, we can apply the Integral Test. $\displaystyle\int_3^\infty \frac{dx}{x\ln x\,(\ln\ln x)^p} = \lim_{t\to\infty}\left[\frac{(\ln\ln x)^{-p+1}}{-p+1}\right]_3^t$ (for $p \neq 1$; if $p = 1$ see Exercise 24) and $\displaystyle\lim_{t\to\infty}\frac{(\ln\ln t)^{-p+1}}{-p+1}$ exists whenever $-p + 1 < 0 \Leftrightarrow p > 1$, so the series converges for $p > 1$.