14. $\dfrac{\sqrt{n}}{n-1} > \dfrac{\sqrt{n}}{n} = \dfrac{1}{\sqrt{n}}$, so $\displaystyle\sum_{n=2}^{\infty} \dfrac{\sqrt{n}}{n-1}$ diverges by comparison with the divergent (partial) p-series $\displaystyle\sum_{n=2}^{\infty} \dfrac{1}{\sqrt{n}}$

$\left(p = \frac{1}{2} \le 1\right)$.

15. $\dfrac{2+(-1)^n}{n\sqrt{n}} \le \dfrac{3}{n\sqrt{n}}$, and $\displaystyle\sum_{n=1}^{\infty} \dfrac{3}{n\sqrt{n}}$ converges because it is a constant multiple of the convergent p-series

$\displaystyle\sum_{n=1}^{\infty} \dfrac{1}{n\sqrt{n}}$ $\left(p = \frac{3}{2} > 1\right)$, so the given series converges by the Comparison Test.

16. $\dfrac{1}{\sqrt{n^3+1}} < \dfrac{1}{\sqrt{n^3}} = \dfrac{1}{n^{3/2}}$, so $\displaystyle\sum_{n=1}^{\infty} \dfrac{1}{\sqrt{n^3+1}}$ converges by comparison with the convergent p-series $\displaystyle\sum_{n=1}^{\infty} \dfrac{1}{n^{3/2}}$

$\left(p = \frac{3}{2} > 1\right)$.

17. Use the Limit Comparison Test with $a_n = \dfrac{1}{\sqrt{n^2+1}}$ and $b_n = \dfrac{1}{n}$:

$\displaystyle\lim_{n\to\infty} \dfrac{a_n}{b_n} = \lim_{n\to\infty} \dfrac{n}{\sqrt{n^2+1}} = \lim_{n\to\infty} \dfrac{1}{\sqrt{1+(1/n^2)}} = 1 > 0$. Since the harmonic series $\displaystyle\sum_{n=1}^{\infty} \dfrac{1}{n}$ diverges, so does

$\displaystyle\sum_{n=1}^{\infty} \dfrac{1}{\sqrt{n^2+1}}$.

18. Use the Limit Comparison Test with $a_n = \dfrac{1}{2n+3}$ and $b_n = \dfrac{1}{n}$:

$\displaystyle\lim_{n\to\infty} \dfrac{a_n}{b_n} = \lim_{n\to\infty} \dfrac{n}{2n+3} = \lim_{n\to\infty} \dfrac{1}{2+(3/n)} = \dfrac{1}{2} > 0$. Since the harmonic series $\displaystyle\sum_{n=1}^{\infty} \dfrac{1}{n}$ diverges, so does

$\displaystyle\sum_{n=1}^{\infty} \dfrac{1}{2n+3}$.

19. $\dfrac{2^n}{1+3^n} < \dfrac{2^n}{3^n} = \left(\dfrac{2}{3}\right)^n$. $\displaystyle\sum_{n=1}^{\infty} \left(\dfrac{2}{3}\right)^n$ is a convergent geometric series ($|r| = \frac{2}{3} < 1$), so $\displaystyle\sum_{n=1}^{\infty} \dfrac{2^n}{1+3^n}$ converges by

the Comparison Test.

20. Use the Limit Comparison Test with $a_n = \dfrac{1+2^n}{1+3^n}$ and $b_n = \dfrac{2^n}{3^n}$: $\displaystyle\lim_{n\to\infty} \dfrac{a_n}{b_n} = \lim_{n\to\infty} \dfrac{(1/2)^n+1}{(1/3)^n+1} = 1 > 0$. Since

$\displaystyle\sum_{n=1}^{\infty} b_n$ converges (geometric series with $|r| = \frac{2}{3} < 1$), $\displaystyle\sum_{n=1}^{\infty} \dfrac{1+2^n}{1+3^n}$ also converges.

21. Use the Limit Comparison Test with $a_n = \dfrac{1}{1+\sqrt{n}}$ and $b_n = \dfrac{1}{\sqrt{n}}$: $\displaystyle\lim_{n\to\infty} \dfrac{a_n}{b_n} = \lim_{n\to\infty} \dfrac{\sqrt{n}}{1+\sqrt{n}} = 1 > 0$. Since

$\displaystyle\sum_{n=1}^{\infty} \dfrac{1}{\sqrt{n}}$ is a divergent p-series ($p = \frac{1}{2} \le 1$), $\displaystyle\sum_{n=1}^{\infty} \dfrac{1}{1+\sqrt{n}}$ also diverges.

22. Use the Limit Comparison Test with $a_n = \dfrac{n+2}{(n+1)^3}$ and $b_n = \dfrac{1}{n^2}$:

$\displaystyle\lim_{n\to\infty} \dfrac{a_n}{b_n} = \lim_{n\to\infty} \dfrac{n^2(n+2)}{(n+1)^3} = \lim_{n\to\infty} \dfrac{1+\frac{2}{n}}{\left(1+\frac{1}{n}\right)^3} = 1 > 0$. Since $\displaystyle\sum_{n=3}^{\infty} \dfrac{1}{n^2}$ is a convergent (partial) p-series

$(p = 2 > 1)$, the series $\displaystyle\sum_{n=3}^{\infty} \dfrac{n+2}{(n+1)^3}$ also converges.

23. Use the Limit Comparison Test with $a_n = \dfrac{5+2n}{(1+n^2)^2}$ and $b_n = \dfrac{1}{n^3}$:

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{n^3(5+2n)}{(1+n^2)^2} = \lim_{n\to\infty}\frac{5n^3+2n^4}{(1+n^2)^2}\cdot\frac{1/n^4}{1/(n^2)^2} = \lim_{n\to\infty}\frac{\frac{5}{n}+2}{\left(\frac{1}{n^2}+1\right)^2} = 2 > 0.\ \text{Since }\sum_{n=1}^{\infty}\frac{1}{n^3}\text{ is a}$$

convergent p-series ($p = 3 > 1$), the series $\displaystyle\sum_{n=1}^{\infty}\frac{5+2n}{(1+n^2)^2}$ also converges.

24. If $a_n = \dfrac{n^2-5n}{n^3+n+1}$ and $b_n = \dfrac{1}{n}$, then $\displaystyle\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{n^3-5n^2}{n^3+n+1} = \lim_{n\to\infty}\frac{1-5/n}{1+1/n^2+1/n^3} = 1 > 0$, so

$\displaystyle\sum_{n=1}^{\infty}\frac{n^2-5n}{n^3+n+1}$ diverges by the Limit Comparison Test with the divergent harmonic series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$. (Note that $a_n > 0$ for $n \geq 6$.)

25. If $a_n = \dfrac{1+n+n^2}{\sqrt{1+n^2+n^6}}$ and $b_n = \dfrac{1}{n}$, then

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{n+n^2+n^3}{\sqrt{1+n^2+n^6}} = \lim_{n\to\infty}\frac{1/n^2+1/n+1}{\sqrt{1/n^6+1/n^4+1}} = 1 > 0,\ \text{so }\sum_{n=1}^{\infty}\frac{1+n+n^2}{\sqrt{1+n^2+n^6}}\text{ diverges by the}$$

Limit Comparison Test with the divergent harmonic series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$.

26. If $a_n = \dfrac{n+5}{\sqrt[3]{n^7+n^2}}$ and $b_n = \dfrac{n}{\sqrt[3]{n^7}} = \dfrac{n}{n^{7/3}} = \dfrac{1}{n^{4/3}}$, then

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{n^{7/3}+5n^{4/3}}{(n^7+n^2)^{1/3}}\cdot\frac{n^{-7/3}}{n^{-7/3}} = \lim_{n\to\infty}\frac{1+5/n}{\left[(n^7+n^2)/n^7\right]^{1/3}}$$

$$= \lim_{n\to\infty}\frac{1+5/n}{(1+1/n^5)^{1/3}} = \frac{1+0}{(1+0)^{1/3}} = 1 > 0,$$

so $\displaystyle\sum_{n=1}^{\infty}\frac{n+5}{\sqrt[3]{n^7+n^2}}$ converges by the Limit Comparison Test with the convergent p-series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^{4/3}}$.

27. Use the Limit Comparison Test with $a_n = \left(1+\dfrac{1}{n}\right)^2 e^{-n}$ and $b_n = e^{-n}$: $\displaystyle\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^2 = 1 > 0.$

Since $\displaystyle\sum_{n=1}^{\infty}e^{-n} = \sum_{n=1}^{\infty}\frac{1}{e^n}$ is a convergent geometric series $\left(|r| = \frac{1}{e} < 1\right)$, the series $\displaystyle\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^2 e^{-n}$ also converges.

28. Use the Limit Comparison Test with $a_n = \dfrac{2n^2+7n}{3^n(n^2+5n-1)}$ and $b_n = \dfrac{1}{3^n}$.

$\displaystyle\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{2n^2+7n}{n^2+5n-1} = 2 > 0$, and since $\displaystyle\sum_{n=1}^{\infty}b_n$ is a convergent geometric series $\left(|r| = \frac{1}{3} < 1\right)$,

$\displaystyle\sum_{n=1}^{\infty}\frac{2n^2+7n}{3^n(n^2+5n-1)}$ converges also.

29. Clearly $n! = n(n-1)(n-2)\cdots(3)(2) \geq 2\cdot2\cdot2\cdots\cdots2\cdot2 = 2^{n-1}$, so $\dfrac{1}{n!} \leq \dfrac{1}{2^{n-1}}.\ \displaystyle\sum_{n=1}^{\infty}\frac{1}{2^{n-1}}$ is a convergent

geometric series $\left(|r| = \frac{1}{2} < 1\right)$, so $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n!}$ converges by the Comparison Test.

30. $\dfrac{n!}{n^n} = \dfrac{1\cdot2\cdot3\cdot\cdots\cdot(n-1)n}{n\cdot n\cdot n\cdot\cdots\cdot n\cdot n} \leq \dfrac{1}{n}\cdot\dfrac{2}{n}\cdot1\cdot1\cdots\cdots1$ for $n \geq 2$, so since $\displaystyle\sum_{n=1}^{\infty}\frac{2}{n^2}$ converges ($p = 2 > 1$), $\displaystyle\sum_{n=1}^{\infty}\frac{n!}{n^n}$

converges also by the Comparison Test.