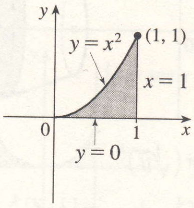
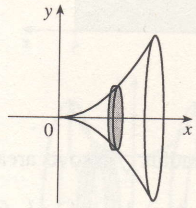
6.2 Volumes

1. A cross-section is circular with radius x^2, so its area is $A(x) = \pi(x^2)^2$.

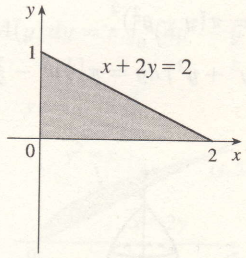
$$V = \int_0^1 A(x)\,dx = \int_0^1 \pi(x^2)^2\,dx = \pi\int_0^1 x^4\,dx = \pi\left[\tfrac{1}{5}x^5\right]_0^1 = \tfrac{\pi}{5}$$

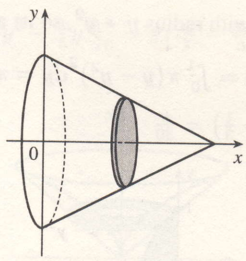




2. $x + 2y = 2 \;\Leftrightarrow\; y = 1 - \frac{1}{2}x$, so a cross-section is circular with radius $1 - \frac{1}{2}x$, and its area is $A(x) = \pi\left(1 - \frac{1}{2}x\right)^2$.

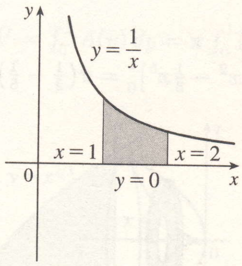
$$V = \int_0^2 \pi y^2\,dx = \pi\int_0^2 \left(1 - \tfrac{1}{2}x\right)^2 dx = \pi\int_0^2 \left(1 - x + \tfrac{1}{4}x^2\right) dx = \pi\left[x - \tfrac{1}{2}x^2 + \tfrac{1}{12}x^3\right]_0^2$$
$$= \pi\left(2 - 2 + \tfrac{2}{3}\right) = \tfrac{2}{3}\pi$$

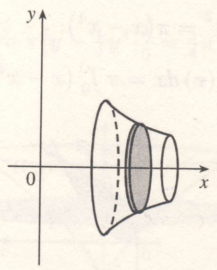




3. A cross-section is a disk with radius $1/x$, so its area is $A(x) = \pi(1/x)^2$.

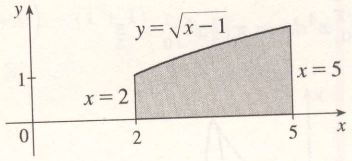
$$V = \int_1^2 A(x)\,dx = \int_1^2 \pi\left(\frac{1}{x}\right)^2 dx = \pi\int_1^2 \frac{1}{x^2}\,dx = \pi\left[-\frac{1}{x}\right]_1^2 = \pi\left[-\tfrac{1}{2} - (-1)\right] = \tfrac{\pi}{2}$$

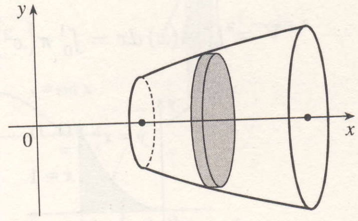




4. A cross-section is circular with radius $\sqrt{x-1}$, so its area is $A(x)=\pi\left(\sqrt{x-1}\right)^2=\pi(x-1)$.

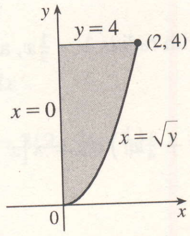
$$V=\int_2^5 A(x)\,dx=\int_2^5 \pi(x-1)\,dx=\pi\left[\tfrac{1}{2}x^2-x\right]_2^5=\pi\left(\tfrac{25}{2}-5-\tfrac{4}{2}+2\right)=\tfrac{15}{2}\pi$$

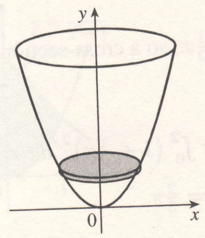




5. A cross-section is a disk with radius $\sqrt{y}$, so its area is $A(y)=\pi\left(\sqrt{y}\right)^2$.

$$V=\int_0^4 A(y)\,dy=\int_0^4 \pi\left(\sqrt{y}\right)^2 dy=\pi\int_0^4 y\,dy=\pi\left[\tfrac{1}{2}y^2\right]_0^4=8\pi$$

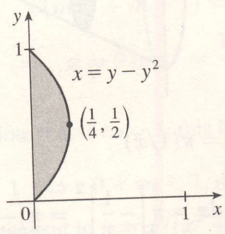


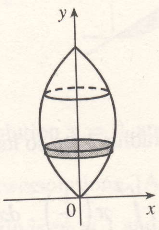


6. A cross-section is a disk with radius $y-y^2$, so its area is $A(y)=\pi\left(y-y^2\right)^2$.

$$V=\int_0^1 A(y)\,dy=\int_0^1 \pi\left(y-y^2\right)^2 dy=\pi\int_0^1\left(y^4-2y^3+y^2\right)dy=\pi\left[\tfrac{1}{5}y^5-\tfrac{1}{2}y^4+\tfrac{1}{3}y^3\right]_0^1$$

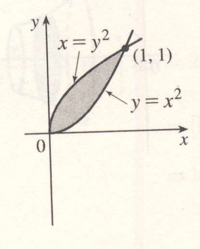
$$=\pi\left(\tfrac{1}{5}-\tfrac{1}{2}+\tfrac{1}{3}\right)=\tfrac{\pi}{30}$$

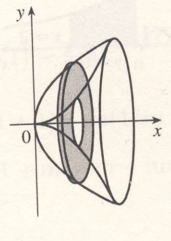




7. A cross-section is a washer (annulus) with inner radius x^2 and outer radius $\sqrt{x}$, so its area is $A(x)=\pi\left(\sqrt{x}\right)^2-\pi\left(x^2\right)^2=\pi\left(x-x^4\right)$.

$$V=\int_0^1 A(x)\,dx=\pi\int_0^1\left(x-x^4\right)dx=\pi\left[\tfrac{1}{2}x^2-\tfrac{1}{5}x^5\right]_0^1=\pi\left(\tfrac{1}{2}-\tfrac{1}{5}\right)=\tfrac{3\pi}{10}$$



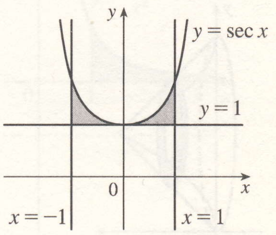


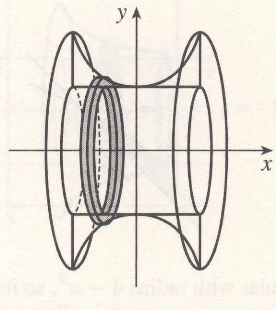
8. A cross-section is a washer with inner radius 1 and outer radius $\sec x$, so its area is

$A(x) = \pi(\sec x)^2 - \pi(1)^2 = \pi(\sec^2 x - 1).$

$$V = \int_{-1}^{1} A(x)\,dx = \int_{-1}^{1} \pi(\sec^2 x - 1)\,dx = 2\pi \int_0^1 (\sec^2 x - 1)\,dx = 2\pi\big[\tan x - x\big]_0^1 = 2\pi(\tan 1 - 1)$$

≈ 3.5023

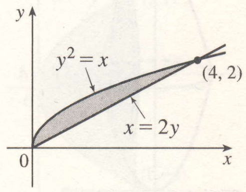


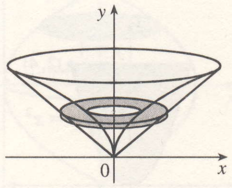


9. A cross-section is a washer with inner radius y^2 and outer radius $2y$, so its area is

$A(y) = \pi(2y)^2 - \pi\left(y^2\right)^2 = \pi(4y^2 - y^4).$

$$V = \int_0^2 A(y)\,dy = \pi \int_0^2 (4y^2 - y^4)\,dy = \pi\left[\tfrac{4}{3}y^3 - \tfrac{1}{5}y^5\right]_0^2 = \pi\left(\tfrac{32}{3} - \tfrac{32}{5}\right) = \tfrac{64\pi}{15}$$

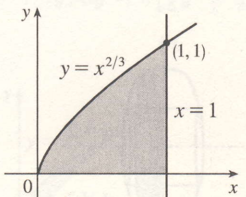


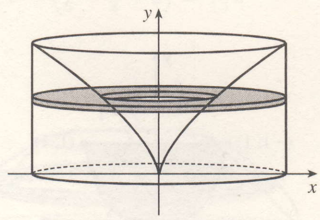


10. $y = x^{2/3} \quad \Leftrightarrow \quad x = y^{3/2}$, so a cross-section is a washer with inner radius $y^{3/2}$ and outer radius 1, and its area is

$A(y) = \pi(1)^2 - \pi\left(y^{3/2}\right)^2 = \pi(1 - y^3).$

$$V = \int_0^1 A(y)\,dy = \pi \int_0^1 (1 - y^3)\,dy = \pi\left[y - \tfrac{1}{4}y^4\right]_0^1 = \tfrac{3}{4}\pi$$

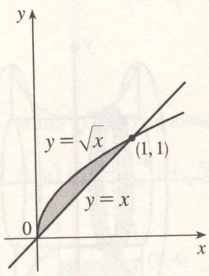


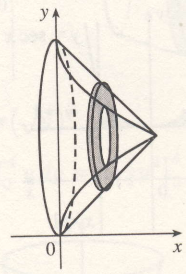


11. A cross-section is a washer with inner radius $1-\sqrt{x}$ and outer radius $1-x$, so its area is

$$A(x)=\pi(1-x)^2-\pi(1-\sqrt{x})^2=\pi\left[(1-2x+x^2)-(1-2\sqrt{x}+x)\right]=\pi\left(-3x+x^2+2\sqrt{x}\right).$$

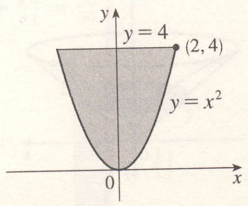
$$V=\int_0^1 A(x)\,dx=\pi\int_0^1\left(-3x+x^2+2\sqrt{x}\right)dx$$
$$=\pi\left[-\tfrac{3}{2}x^2+\tfrac{1}{3}x^3+\tfrac{4}{3}x^{3/2}\right]_0^1=\pi\left(-\tfrac{3}{2}+\tfrac{5}{3}\right)=\tfrac{\pi}{6}$$

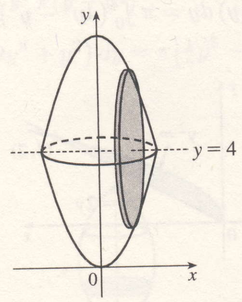




12. A cross-section is circular with radius $4-x^2$, so its area is $A(x)=\pi(4-x^2)^2=\pi(16-8x^2+x^4)$.

$$V=\int_{-2}^2 A(x)\,dx=2\int_0^2 A(x)\,dx=2\pi\int_0^2\left(16-8x^2+x^4\right)dx=2\pi\left[16x-\tfrac{8}{3}x^3+\tfrac{1}{5}x^5\right]_0^2$$
$$=2\pi\left(32-\tfrac{64}{3}+\tfrac{32}{5}\right)=64\pi\left(1-\tfrac{2}{3}+\tfrac{1}{5}\right)=64\pi\cdot\tfrac{8}{15}=\tfrac{512\pi}{15}$$

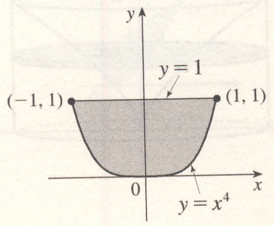


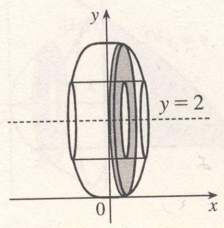


13. A cross-section is an annulus with inner radius $2-1$ and outer radius $2-x^4$, so its area is

$$A(x)=\pi(2-x^4)^2-\pi(2-1)^2=\pi(3-4x^4+x^8).$$

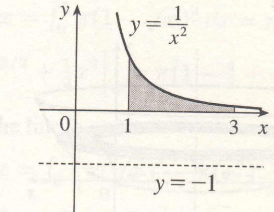
$$V=\int_{-1}^1 A(x)\,dx=2\int_0^1 A(x)\,dx=2\pi\int_0^1\left(3-4x^4+x^8\right)dx=2\pi\left[3x-\tfrac{4}{5}x^5+\tfrac{1}{9}x^9\right]_0^1$$
$$=2\pi\left(3-\tfrac{4}{5}+\tfrac{1}{9}\right)=\tfrac{208}{45}\pi$$

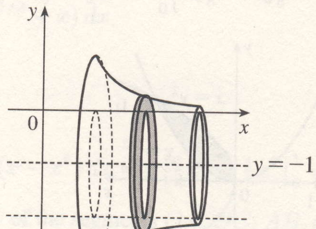




14. $V = \displaystyle\int_1^3 \pi \left\{ \left[\frac{1}{x^2} - (-1) \right]^2 - [0 - (-1)]^2 \right\} dx = \pi \int_1^3 \left[\left(\frac{1}{x^2} + 1 \right)^2 - 1^2 \right] dx = \pi \int_1^3 \left(\frac{1}{x^4} + \frac{2}{x^2} \right) dx$

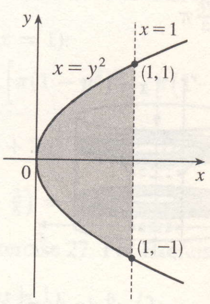
$= \pi \left[-\dfrac{1}{3x^3} - \dfrac{2}{x} \right]_1^3 = \pi\left[\left(-\frac{1}{81} - \frac{2}{3}\right) - \left(-\frac{1}{3} - 2\right)\right] = \frac{134\pi}{81}$

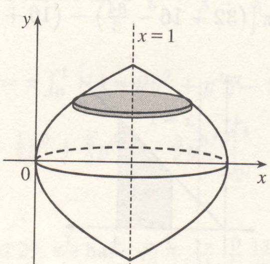




15. $V = \int_{-1}^1 \pi(1 - y^2)^2\, dy = 2\int_0^1 \pi(1 - y^2)^2\, dy = 2\pi \int_0^1 (1 - 2y^2 + y^4)\, dy$

$= 2\pi\left[y - \frac{2}{3}y^3 + \frac{1}{5}y^5\right]_0^1 = 2\pi \cdot \frac{8}{15} = \frac{16}{15}\pi$





16. $y = \sqrt{x} \;\Rightarrow\; x = y^2$, so the outer radius is $2 - y^2$.

$V = \int_0^1 \pi\left[(2 - y^2)^2 - (2 - y)^2\right] dy = \pi \int_0^1 \left[(4 - 4y^2 + y^4) - (4 - 4y + y^2)\right] dy$

$= \pi \int_0^1 (y^4 - 5y^2 + 4y)\, dy = \pi\left[\frac{1}{5}y^5 - \frac{5}{3}y^3 + 2y^2\right]_0^1 = \pi\left(\frac{1}{5} - \frac{5}{3} + 2\right) = \frac{8}{15}\pi$