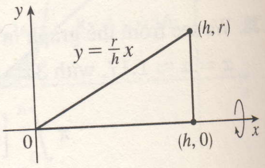
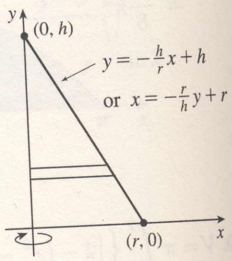
46. $V = \int_0^{10} A(x)\,dx \approx M_5 = \frac{10-0}{5}[A(1) + A(3) + A(5) + A(7) + A(9)]$
$= 2(0.65 + 0.61 + 0.59 + 0.55 + 0.50) = 2(2.90) = 5.80\ \text{m}^3$

47. We'll form a right circular cone with height h and base radius r by revolving the line $y = \frac{r}{h}x$ about the x-axis.



$$V = \pi \int_0^h \left(\frac{r}{h}x\right)^2 dx = \pi \int_0^h \frac{r^2}{h^2} x^2 dx = \pi \frac{r^2}{h^2}\left[\frac{1}{3}x^3\right]_0^h$$
$$= \pi \frac{r^2}{h^2}\left(\frac{1}{3}h^3\right) = \frac{1}{3}\pi r^2 h$$

Another solution: Revolve $x = -\frac{r}{h}y + r$ about the y-axis.

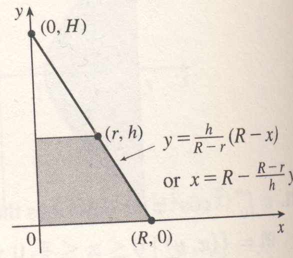


$$V = \pi \int_0^h \left(-\frac{r}{h}y + r\right)^2 dy \overset{*}{=} \pi \int_0^h \left[\frac{r^2}{h^2}y^2 - \frac{2r^2}{h}y + r^2\right] dy$$
$$= \pi\left[\frac{r^2}{3h^2}y^3 - \frac{r^2}{h}y^2 + r^2 y\right]_0^h = \pi\left(\tfrac{1}{3}r^2h - r^2h + r^2h\right) = \tfrac{1}{3}\pi r^2 h$$

* Or use substitution with $u = r - \frac{r}{h}y$ and $du = -\frac{r}{h}\,dy$ to get

$$\pi \int_r^0 u^2\left(-\frac{h}{r}\,du\right) = -\pi\frac{h}{r}\left[\frac{1}{3}u^3\right]_r^0 = -\pi\frac{h}{r}\left(-\frac{1}{3}r^3\right) = \frac{1}{3}\pi r^2 h.$$

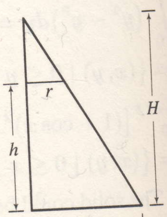
48. $$V = \pi \int_0^h \left(R - \frac{R-r}{h}y\right)^2 dy$$



$$= \pi \int_0^h \left[R^2 - \frac{2R(R-r)}{h}y + \left(\frac{R-r}{h}\right)^2 y^2\right] dy$$
$$= \pi\left[R^2 y - \frac{R(R-r)}{h}y^2 + \frac{1}{3}\left(\frac{R-r}{h}\right)^2 y^3\right]_0^h$$
$$= \pi\left[R^2h - R(R-r)h + \tfrac{1}{3}(R-r)^2h\right]$$
$$= \tfrac{1}{3}\pi h\left[3Rr + (R^2 - 2Rr + r^2)\right] = \tfrac{1}{3}\pi h(R^2 + Rr + r^2)$$

Another solution: $\frac{H}{R} = \frac{H-h}{r}$ by similar triangles. Therefore,

$Hr = HR - hR \Rightarrow hR = H(R-r) \Rightarrow H = \frac{hR}{R-r}$. Now



$$V = \tfrac{1}{3}\pi R^2 H - \tfrac{1}{3}\pi r^2(H-h) \quad \text{[by Exercise 47]}$$
$$= \tfrac{1}{3}\pi R^2 \frac{hR}{R-r} - \tfrac{1}{3}\pi r^2 \frac{rh}{R-r} \quad \left[H - h = \frac{rH}{R} = \frac{rhR}{R(R-r)}\right]$$
$$= \frac{1}{3}\pi h\frac{R^3 - r^3}{R-r} = \tfrac{1}{3}\pi h(R^2 + Rr + r^2)$$
$$= \tfrac{1}{3}\left[\pi R^2 + \pi r^2 + \sqrt{(\pi R^2)(\pi r^2)}\right]h = \tfrac{1}{3}\left(A_1 + A_2 + \sqrt{A_1 A_2}\right)h$$

where A_1 and A_2 are the areas of the bases of the frustum. (See Exercise 50 for a related result.)

49. $x^2 + y^2 = r^2 \quad \Leftrightarrow \quad x^2 = r^2 - y^2$

$$V = \pi \int_{r-h}^{r} (r^2 - y^2)\,dy = \pi \left[r^2 y - \frac{y^3}{3} \right]_{r-h}^{r}$$

$$= \pi \left\{ \left[r^3 - \frac{r^3}{3} \right] - \left[r^2(r-h) - \frac{(r-h)^3}{3} \right] \right\}$$

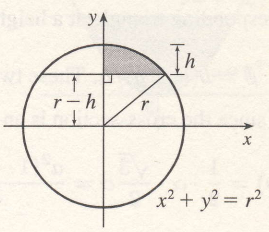
$$= \pi \left\{ \tfrac{2}{3} r^3 - \tfrac{1}{3}(r-h)\left[3r^2 - (r-h)^2\right] \right\}$$

$$= \tfrac{1}{3}\pi \left\{ 2r^3 - (r-h)\left[3r^2 - (r^2 - 2rh + h^2)\right] \right\}$$

$$= \tfrac{1}{3}\pi \left\{ 2r^3 - (r-h)\left[2r^2 + 2rh - h^2\right] \right\}$$

$$= \tfrac{1}{3}\pi \left(2r^3 - 2r^3 - 2r^2 h + rh^2 + 2r^2 h + 2rh^2 - h^3 \right)$$

$$= \tfrac{1}{3}\pi (3rh^2 - h^3) = \tfrac{1}{3}\pi h^2 (3r - h), \text{ or, equivalently, } \pi h^2 \left(r - \frac{h}{3} \right)$$



50. An equation of the line is $x = \dfrac{\Delta x}{\Delta y} y + (x\text{-intercept}) = \dfrac{a/2 - b/2}{h - 0} y + \dfrac{b}{2} = \dfrac{a-b}{2h} y + \dfrac{b}{2}$.

$$V = \int_0^h A(y)\,dy = \int_0^h (2x)^2\,dy$$

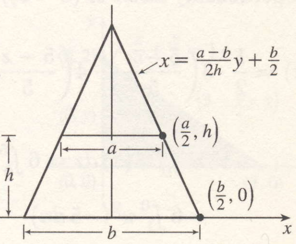
$$= \int_0^h \left[2\left(\frac{a-b}{2h} y + \frac{b}{2} \right) \right]^2 dy = \int_0^h \left[\frac{a-b}{h} y + b \right]^2 dy$$

$$= \int_0^h \left[\frac{(a-b)^2}{h^2} y^2 + \frac{2b(a-b)}{h} y + b^2 \right] dy$$

$$= \left[\frac{(a-b)^2}{3h^2} y^3 + \frac{b(a-b)}{h} y^2 + b^2 y \right]_0^h$$

$$= \tfrac{1}{3}(a-b)^2 h + b(a-b)h + b^2 h = \tfrac{1}{3}(a^2 - 2ab + b^2 + 3ab)h$$

$$= \tfrac{1}{3}(a^2 + ab + b^2)h$$



[Note that this can be written as $\frac{1}{3}\left(A_1 + A_2 + \sqrt{A_1 A_2}\right)h$, as in Exercise 48.]

If $a = b$, we get a rectangular solid with volume $b^2 h$. If $a = 0$, we get a square pyramid with volume $\frac{1}{3} b^2 h$.

51. For a cross-section at height y, we see from similar triangles that $\dfrac{\alpha/2}{b/2} = \dfrac{h-y}{h}$, so $\alpha = b\left(1 - \dfrac{y}{h}\right)$.

Similarly, for cross-sections having $2b$ as their base and β replacing α, $\beta = 2b\left(1 - \dfrac{y}{h}\right)$. So

$$V = \int_0^h A(y)\,dy = \int_0^h \left[b\left(1 - \tfrac{y}{h}\right) \right] \left[2b\left(1 - \tfrac{y}{h}\right) \right] dy$$

$$= \int_0^h 2b^2 \left(1 - \tfrac{y}{h}\right)^2 dy = 2b^2 \int_0^h \left(1 - \frac{2y}{h} + \frac{y^2}{h^2} \right) dy$$

$$= 2b^2 \left[y - \frac{y^2}{h} + \frac{y^3}{3h^2} \right]_0^h = 2b^2 \left[h - h + \tfrac{1}{3} h \right]$$

$$= \tfrac{2}{3} b^2 h \quad \left[= \tfrac{1}{3} Bh \text{ where } B \text{ is the area of the base, as with any pyramid.} \right]$$