55. Let $u = \sqrt{x+1}$. Then $x = u^2 - 1 \quad\Rightarrow$

$$\int \frac{dx}{x+4+4\sqrt{x+1}} = \int \frac{2u\,du}{u^2+3+4u} = \int \left[\frac{-1}{u+1} + \frac{3}{u+3}\right] du$$
$$= 3\ln|u+3| - \ln|u+1| + C = 3\ln\left(\sqrt{x+1}+3\right) - \ln\left(\sqrt{x+1}+1\right) + C$$

56. Let $t = \sqrt{x^2-1}$. Then $dt = \left(x/\sqrt{x^2-1}\right)dx$, $x^2 - 1 = t^2$, $x = \sqrt{t^2+1}$, so

$I = \displaystyle\int \frac{x\ln x}{\sqrt{x^2-1}}\,dx = \int \ln\sqrt{t^2+1}\,dt = \frac{1}{2}\int \ln\left(t^2+1\right)dt$. Now use parts with $u = \ln\left(t^2+1\right)$, $dv = dt$:

$$I = \tfrac{1}{2}t\ln\left(t^2+1\right) - \int \frac{t^2}{t^2+1}\,dt = \tfrac{1}{2}t\ln\left(t^2+1\right) - \int \left[1 - \frac{1}{t^2+1}\right] dt$$
$$= \tfrac{1}{2}t\ln\left(t^2+1\right) - t + \tan^{-1}t + C = \sqrt{x^2-1}\,\ln x - \sqrt{x^2-1} + \tan^{-1}\sqrt{x^2-1} + C$$

Another method: First integrate by parts with $u = \ln x$, $dv = \left(x/\sqrt{x^2-1}\right)dx$ and then use substitution $\left(x = \sec\theta \text{ or } u = \sqrt{x^2-1}\right)$.

57. Let $u = \sqrt[3]{x+c}$. Then $x = u^3 - c \quad\Rightarrow$

$$\textstyle\int x\sqrt[3]{x+c}\,dx = \int \left(u^3 - c\right)u \cdot 3u^2\,du = 3\int \left(u^6 - cu^3\right)du = \frac{3}{7}u^7 - \frac{3}{4}cu^4 + C$$
$$= \tfrac{3}{7}(x+c)^{7/3} - \tfrac{3}{4}c(x+c)^{4/3} + C$$

58. Integrate by parts with $u = \ln(1+x)$, $dv = x^2\,dx \quad\Rightarrow\quad du = dx/(1+x)$, $v = \frac{1}{3}x^3$:

$$\int x^2\ln(1+x)\,dx = \tfrac{1}{3}x^3\ln(1+x) - \int \frac{x^3\,dx}{3(1+x)} = \tfrac{1}{3}x^3\ln(1+x) - \frac{1}{3}\int \left(x^2 - x + 1 - \frac{1}{x+1}\right)dx$$
$$= \tfrac{1}{3}x^3\ln(1+x) - \tfrac{1}{9}x^3 + \tfrac{1}{6}x^2 - \tfrac{1}{3}x + \tfrac{1}{3}\ln(1+x) + C$$

59. Let $u = e^x$. Then $x = \ln u$, $dx = du/u \quad\Rightarrow$

$$\int \frac{dx}{e^{3x}-e^x} = \int \frac{du/u}{u^3-u} = \int \frac{du}{(u-1)u^2(u+1)} = \int \left[\frac{1/2}{u-1} - \frac{1}{u^2} - \frac{1/2}{u+1}\right] du$$
$$= \frac{1}{u} + \frac{1}{2}\ln\left|\frac{u-1}{u+1}\right| + C = e^{-x} + \frac{1}{2}\ln\left|\frac{e^x-1}{e^x+1}\right| + C$$

60. Let $u = \sqrt[3]{x}$. Then $x = u^3$, $dx = 3u^2\,du \quad\Rightarrow$

$$\int \frac{dx}{x+\sqrt[3]{x}} = \int \frac{3u^2\,du}{u^3+u} = \frac{3}{2}\int \frac{2u\,du}{u^2+1} = \tfrac{3}{2}\ln\left(u^2+1\right) + C = \tfrac{3}{2}\ln\left(x^{2/3}+1\right) + C.$$

61. Let $u = x^5$. Then $du = 5x^4\,dx \quad\Rightarrow$

$$\int \frac{x^4\,dx}{x^{10}+16} = \int \frac{\frac{1}{5}\,du}{u^2+16} = \tfrac{1}{5}\cdot\tfrac{1}{4}\tan^{-1}\left(\tfrac{1}{4}u\right) + C = \tfrac{1}{20}\tan^{-1}\left(\tfrac{1}{4}x^5\right) + C.$$

62. Let $u = x+1$. Then $du = dx \quad\Rightarrow$

$$\int \frac{x^3}{(x+1)^{10}}\,dx = \int \frac{(u-1)^3}{u^{10}}\,du = \int \left(u^{-7} - 3u^{-8} + 3u^{-9} - u^{-10}\right)du$$
$$= -\tfrac{1}{6}u^{-6} + \tfrac{3}{7}u^{-7} - \tfrac{3}{8}u^{-8} + \tfrac{1}{9}u^{-9} + C$$
$$= (x+1)^{-9}\left[-\tfrac{1}{6}(x+1)^3 + \tfrac{3}{7}(x+1)^2 - \tfrac{3}{8}(x+1) + \tfrac{1}{9}\right] + C$$

63. Let $y = \sqrt{x}$ so that $dy = \dfrac{1}{2\sqrt{x}}\,dx \;\Rightarrow\; dx = 2\sqrt{x}\,dy = 2y\,dy$. Then

$$\begin{aligned}
\int \sqrt{x}\,e^{\sqrt{x}}\,dx &= \int ye^y(2y\,dy) = \int 2y^2e^y\,dy \qquad \begin{bmatrix} u = 2y^2, & dv = e^y\,dy, \\ du = 4y\,dy & v = e^y \end{bmatrix} \\
&= 2y^2e^y - \int 4ye^y\,dy \qquad \begin{bmatrix} U = 4y, & dV = e^y\,dy, \\ dU = 4\,dy & V = e^y \end{bmatrix} \\
&= 2y^2e^y - \left(4ye^y - \textstyle\int 4e^y\,dy\right) = 2y^2e^y - 4ye^y + 4e^y + C \\
&= 2(y^2 - 2y + 2)e^y + C = 2(x - 2\sqrt{x} + 2)\,e^{\sqrt{x}} + C
\end{aligned}$$

64. Let $u = \tan x$. Then

$$\begin{aligned}
\int_{\pi/4}^{\pi/3} \frac{\ln(\tan x)\,dx}{\sin x\cos x} &= \int_{\pi/4}^{\pi/3} \frac{\ln(\tan x)}{\tan x}\sec^2 x\,dx = \int_1^{\sqrt{3}} \frac{\ln u}{u}\,du \\
&= \left[\tfrac{1}{2}(\ln u)^2\right]_1^{\sqrt{3}} = \tfrac{1}{2}\left(\ln\sqrt{3}\right)^2 = \tfrac{1}{8}(\ln 3)^2
\end{aligned}$$

65.
$$\begin{aligned}
\int \frac{dx}{\sqrt{x+1}+\sqrt{x}} &= \int \left(\frac{1}{\sqrt{x+1}+\sqrt{x}} \cdot \frac{\sqrt{x+1}-\sqrt{x}}{\sqrt{x+1}-\sqrt{x}}\right)dx = \int \left(\sqrt{x+1}-\sqrt{x}\right)dx \\
&= \tfrac{2}{3}\left[(x+1)^{3/2} - x^{3/2}\right] + C
\end{aligned}$$

66. $\displaystyle\int \frac{u^3+1}{u^3-u^2}\,du = \int\left[1 + \frac{u^2+1}{(u-1)u^2}\right]du = u + \int\left[\frac{2}{u-1} - \frac{1}{u} - \frac{1}{u^2}\right]du = u + 2\ln|u-1| - \ln|u| + \frac{1}{u} + C.$

Thus,

$$\begin{aligned}
\int_2^3 \frac{u^3+1}{u^3-u^2}\,du &= \left[u + 2\ln(u-1) - \ln u + \frac{1}{u}\right]_2^3 = \left(3 + 2\ln 2 - \ln 3 + \tfrac{1}{3}\right) - \left(2 + 2\ln 1 - \ln 2 + \tfrac{1}{2}\right) \\
&= 1 + 3\ln 2 - \ln 3 - \tfrac{1}{6} = \tfrac{5}{6} + \ln\tfrac{8}{3}
\end{aligned}$$

67. Let $u = \sqrt{t}$. Then $du = dt/(2\sqrt{t}\,) \;\Rightarrow$

$$\begin{aligned}
\int_1^3 \frac{\arctan\sqrt{t}}{\sqrt{t}}\,dt &= \int_1^{\sqrt{3}} \tan^{-1}u\,(2\,du) = 2\left[u\tan^{-1}u - \tfrac{1}{2}\ln(1+u^2)\right]_1^{\sqrt{3}} \qquad \text{[Example 5 in Section 8.1]} \\
&= 2\left[\left(\sqrt{3}\tan^{-1}\sqrt{3} - \tfrac{1}{2}\ln 4\right) - \left(\tan^{-1}1 - \tfrac{1}{2}\ln 2\right)\right] \\
&= 2\left[\left(\sqrt{3}\cdot\tfrac{\pi}{3} - \ln 2\right) - \left(\tfrac{\pi}{4} - \tfrac{1}{2}\ln 2\right)\right] = \tfrac{2}{3}\sqrt{3}\,\pi - \tfrac{1}{2}\pi - \ln 2
\end{aligned}$$

68. Let $u = e^x$. Then $x = \ln u$, $dx = du/u \;\Rightarrow$

$$\begin{aligned}
\int \frac{dx}{1 + 2e^x - e^{-x}} &= \int \frac{du/u}{1 + 2u - 1/u} = \int \frac{du}{2u^2 + u - 1} = \int\left[\frac{2/3}{2u-1} - \frac{1/3}{u+1}\right]du \\
&= \tfrac{1}{3}\ln|2u-1| - \tfrac{1}{3}\ln|u+1| + C = \tfrac{1}{3}\ln\left|(2e^x - 1)/(e^x + 1)\right| + C
\end{aligned}$$

69. Let $u = e^x$. Then $x = \ln u$, $dx = du/u \;\Rightarrow$

$$\begin{aligned}
\int \frac{e^{2x}}{1+e^x}\,dx &= \int \frac{u^2}{1+u}\frac{du}{u} = \int \frac{u}{1+u}\,du = \int\left(1 - \frac{1}{1+u}\right)du \\
&= u - \ln|1+u| + C = e^x - \ln(1+e^x) + C
\end{aligned}$$

70. Use parts with $u = \ln(x+1)$, $dv = dx/x^2$:

$$\begin{aligned}
\int \frac{\ln(x+1)}{x^2}\,dx &= -\frac{1}{x}\ln(x+1) + \int \frac{dx}{x(x+1)} = -\frac{1}{x}\ln(x+1) + \int\left[\frac{1}{x} - \frac{1}{x+1}\right]dx \\
&= -\tfrac{1}{x}\ln(x+1) + \ln|x| - \ln(x+1) + C = -\left(1 + \tfrac{1}{x}\right)\ln(x+1) + \ln|x| + C
\end{aligned}$$

71. $\dfrac{x}{x^4+4x^2+3} = \dfrac{x}{(x^2+3)(x^2+1)} = \dfrac{Ax+B}{x^2+3} + \dfrac{Cx+D}{x^2+1} \Rightarrow$

$$x = (Ax+B)(x^2+1) + (Cx+D)(x^2+3) = (Ax^3+Bx^2+Ax+B) + (Cx^3+Dx^2+3Cx+3D)$$
$$= (A+C)x^3 + (B+D)x^2 + (A+3C)x + (B+3D) \Rightarrow$$

$A+C=0$, $B+D=0$, $A+3C=1$, $B+3D=0 \Rightarrow A=-\frac{1}{2}$, $C=\frac{1}{2}$, $B=0$, $D=0$. Thus,

$$\int \frac{x}{x^4+4x^2+3}\,dx = \int \left(\frac{-\frac{1}{2}x}{x^2+3} + \frac{\frac{1}{2}x}{x^2+1}\right) dx$$
$$= -\tfrac{1}{4}\ln(x^2+3) + \tfrac{1}{4}\ln(x^2+1) + C \text{ or } \frac{1}{4}\ln\left(\frac{x^2+1}{x^2+3}\right) + C$$

72. Let $u = \sqrt[6]{t}$. Then $t = u^6$, $dt = 6u^5\,du \Rightarrow$

$$\int \frac{\sqrt{t}\,dt}{1+\sqrt[3]{t}} = \int \frac{u^3 \cdot 6u^5\,du}{1+u^2} = 6\int \frac{u^8}{u^2+1}\,du = 6\int \left(u^6 - u^4 + u^2 - 1 + \frac{1}{u^2+1}\right) du$$
$$= 6\left(\tfrac{1}{7}u^7 - \tfrac{1}{5}u^5 + \tfrac{1}{3}u^3 - u + \tan^{-1} u\right) + C$$
$$= 6\left(\tfrac{1}{7}t^{7/6} - \tfrac{1}{5}t^{5/6} + \tfrac{1}{3}t^{1/2} - t^{1/6} + \tan^{-1} t^{1/6}\right) + C$$

73. $\dfrac{1}{(x-2)(x^2+4)} = \dfrac{A}{x-2} + \dfrac{Bx+C}{x^2+4} \Rightarrow$

$1 = A(x^2+4) + (Bx+C)(x-2) = (A+B)x^2 + (C-2B)x + (4A-2C)$. So $0 = A+B = C-2B$, $1 = 4A-2C$. Setting $x=2$ gives $A = \frac{1}{8} \Rightarrow B = -\frac{1}{8}$ and $C = -\frac{1}{4}$. So

$$\int \frac{1}{(x-2)(x^2+4)}\,dx = \int \left(\frac{\frac{1}{8}}{x-2} + \frac{-\frac{1}{8}x - \frac{1}{4}}{x^2+4}\right) dx = \frac{1}{8}\int \frac{dx}{x-2} - \frac{1}{16}\int \frac{2x\,dx}{x^2+4} - \frac{1}{4}\int \frac{dx}{x^2+4}$$
$$= \tfrac{1}{8}\ln|x-2| - \tfrac{1}{16}\ln(x^2+4) - \tfrac{1}{8}\tan^{-1}(x/2) + C$$

74. Let $u = e^x$. Then $x = \ln u$, $dx = du/u \Rightarrow$

$$\int \frac{dx}{e^x - e^{-x}} = \int \frac{e^x\,dx}{e^{2x}-1} = \int \frac{u}{u^2-1}\frac{du}{u} = \int \frac{du}{u^2-1} = \frac{1}{2}\ln\left|\frac{u-1}{u+1}\right| + C = \frac{1}{2}\ln\left(\frac{|e^x-1|}{e^x+1}\right) + C.$$

75. $\int \sin x \sin 2x \sin 3x\,dx = \int \sin x \cdot \frac{1}{2}[\cos(2x-3x) - \cos(2x+3x)]\,dx = \frac{1}{2}\int (\sin x \cos x - \sin x \cos 5x)\,dx$

$$= \tfrac{1}{4}\int \sin 2x\,dx - \tfrac{1}{2}\int \tfrac{1}{2}[\sin(x+5x) + \sin(x-5x)]\,dx$$
$$= -\tfrac{1}{8}\cos 2x - \tfrac{1}{4}\int(\sin 6x - \sin 4x)\,dx = -\tfrac{1}{8}\cos 2x + \tfrac{1}{24}\cos 6x - \tfrac{1}{16}\cos 4x + C$$

76. $\int (x^2 - bx)\sin 2x\,dx = -\frac{1}{2}(x^2-bx)\cos 2x + \frac{1}{2}\int(2x-b)\cos 2x\,dx$

$$[u = x^2 - bx,\ dv = \sin 2x\,dx,\ du = (2x-b)\,dx,\ v = -\tfrac{1}{2}\cos 2x]$$
$$= -\tfrac{1}{2}(x^2-bx)\cos 2x + \tfrac{1}{2}\left[\tfrac{1}{2}(2x-b)\sin 2x - \int \sin 2x\,dx\right]$$
$$[U = 2x-b,\ dV = \cos 2x\,dx,\ dU = 2\,dx,\ V = \tfrac{1}{2}\sin 2x]$$
$$= -\tfrac{1}{2}(x^2-bx)\cos 2x + \tfrac{1}{4}(2x-b)\sin 2x + \tfrac{1}{4}\cos 2x + C$$

77. Let $u = x^{3/2}$ so that $u^2 = x^3$ and $du = \frac{3}{2}x^{1/2}\,dx \quad\Rightarrow\quad \sqrt{x}\,dx = \frac{2}{3}\,du$. Then

$$\int \frac{\sqrt{x}}{1+x^3}\,dx = \int \frac{\frac{2}{3}}{1+u^2}\,du = \frac{2}{3}\tan^{-1}u + C = \frac{2}{3}\tan^{-1}(x^{3/2}) + C.$$

78. $$\int \frac{\sec x\,\cos 2x}{\sin x + \sec x}\,dx = \int \frac{\sec x\,\cos 2x}{\sin x + \sec x}\cdot\frac{2\cos x}{2\cos x}\,dx = \int \frac{2\cos 2x}{2\sin x\cos x + 2}\,dx$$

$$= \int \frac{2\cos 2x}{\sin 2x + 2}\,dx = \int \frac{1}{u}\,du \qquad \begin{bmatrix} u = \sin 2x + 2, \\ du = 2\cos 2x\,dx \end{bmatrix}$$

$$= \ln|u| + C = \ln|\sin 2x + 2| + C = \ln(\sin 2x + 2) + C$$

79. Let $u = x$, $dv = \sin^2 x\cos x\,dx \quad\Rightarrow\quad du = dx$, $v = \frac{1}{3}\sin^3 x$. Then

$$\int x\sin^2 x\,\cos x\,dx = \tfrac{1}{3}x\sin^3 x - \int \tfrac{1}{3}\sin^3 x\,dx = \tfrac{1}{3}x\sin^3 x - \tfrac{1}{3}\int (1-\cos^2 x)\sin x\,dx$$

$$= \frac{1}{3}x\sin^3 x + \frac{1}{3}\int (1-y^2)\,dy \qquad \begin{bmatrix} y = \cos x, \\ dy = -\sin x\,dx \end{bmatrix}$$

$$= \tfrac{1}{3}x\sin^3 x + \tfrac{1}{3}y - \tfrac{1}{9}y^3 + C = \tfrac{1}{3}x\sin^3 x + \tfrac{1}{3}\cos x - \tfrac{1}{9}\cos^3 x + C$$

80. $$\int \frac{\sin x\,\cos x}{\sin^4 x + \cos^4 x}\,dx = \int \frac{\sin x\,\cos x}{(\sin^2 x)^2 + (\cos^2 x)^2}\,dx = \int \frac{\sin x\,\cos x}{(\sin^2 x)^2 + (1-\sin^2 x)^2}\,dx$$

$$= \int \frac{1}{u^2 + (1-u)^2}\left(\frac{1}{2}\,du\right) \qquad \begin{bmatrix} u = \sin^2 x, \\ du = 2\sin x\cos x\,dx \end{bmatrix}$$

$$= \int \frac{1}{4u^2 - 4u + 2}\,du = \int \frac{1}{(4u^2 - 4u + 1) + 1}\,du$$

$$= \int \frac{1}{(2u-1)^2 + 1}\,du = \frac{1}{2}\int \frac{1}{y^2+1}\,dy \qquad \begin{bmatrix} y = 2u - 1, \\ dy = 2\,du \end{bmatrix}$$

$$= \tfrac{1}{2}\tan^{-1}y + C = \tfrac{1}{2}\tan^{-1}(2u-1) + C = \tfrac{1}{2}\tan^{-1}(2\sin^2 x - 1) + C$$

Another solution:

$$\int \frac{\sin x\,\cos x}{\sin^4 x + \cos^4 x}\,dx = \int \frac{(\sin x\,\cos x)/\cos^4 x}{(\sin^4 x + \cos^4 x)/\cos^4 x}\,dx = \int \frac{\tan x\sec^2 x}{\tan^4 x + 1}\,dx$$

$$= \int \frac{1}{u^2+1}\left(\frac{1}{2}\,du\right) \qquad \begin{bmatrix} u = \tan^2 x, \\ du = 2\tan x\sec^2 x\,dx \end{bmatrix}$$

$$= \tfrac{1}{2}\tan^{-1}u + C = \tfrac{1}{2}\tan^{-1}(\tan^2 x) + C$$

81. The function $y = 2xe^{x^2}$ *does* have an elementary antiderivative, so we'll use this fact to help evaluate the integral.

$$\int (2x^2+1)e^{x^2}\,dx = \int 2x^2e^{x^2}\,dx + \int e^{x^2}\,dx = \int x(2xe^{x^2})\,dx + \int e^{x^2}\,dx$$

$$= xe^{x^2} - \int e^{x^2}\,dx + \int e^{x^2}\,dx \qquad \begin{bmatrix} u = x, & dv = 2xe^{x^2}\,dx, \\ du = dx & v = e^{x^2} \end{bmatrix} \qquad = xe^{x^2} + C$$