19. $\dfrac{1}{(x+5)^2(x-1)} = \dfrac{A}{x+5} + \dfrac{B}{(x+5)^2} + \dfrac{C}{x-1} \Rightarrow 1 = A(x+5)(x-1) + B(x-1) + C(x+5)^2$. Setting $x=-5$ gives $1=-6B$, so $B=-\frac{1}{6}$. Setting $x=1$ gives $1=36C$, so $C=\frac{1}{36}$. Setting $x=-2$ gives $1=A(3)(-3)+B(-3)+C(3^2)=-9A-3B+9C=-9A+\frac{1}{2}+\frac{1}{4}=-9A+\frac{3}{4}$, so $9A=-\frac{1}{4}$ and $A=-\frac{1}{36}$. Now

$$\int \frac{1}{(x+5)^2(x-1)}\,dx = \int\left[\frac{-1/36}{x+5} - \frac{1/6}{(x+5)^2} + \frac{1/36}{x-1}\right]dx$$
$$= -\frac{1}{36}\ln|x+5| + \frac{1}{6(x+5)} + \frac{1}{36}\ln|x-1| + C$$

20. $\dfrac{x^2}{(x-3)(x+2)^2} = \dfrac{A}{x-3} + \dfrac{B}{x+2} + \dfrac{C}{(x+2)^2} \Rightarrow x^2 = A(x+2)^2 + B(x-3)(x+2) + C(x-3)$.

Setting $x=3$ gives $A=\frac{9}{25}$. Take $x=-2$ to get $C=-\frac{4}{5}$, and equate the coefficients of x^2 to get $1=A+B \Rightarrow B=\frac{16}{25}$. Then

$$\int \frac{x^2}{(x-3)(x+2)^2}\,dx = \int\left[\frac{9/25}{x-3} + \frac{16/25}{x+2} - \frac{4/5}{(x+2)^2}\right]dx$$
$$= \frac{9}{25}\ln|x-3| + \frac{16}{25}\ln|x+2| + \frac{4}{5(x+2)} + C$$

21. $\dfrac{5x^2+3x-2}{x^3+2x^2} = \dfrac{5x^2+3x-2}{x^2(x+2)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x+2}$. Multiply by $x^2(x+2)$ to get $5x^2+3x-2 = Ax(x+2) + B(x+2) + Cx^2$. Set $x=-2$ to get $C=3$, and take $x=0$ to get $B=-1$. Equating the coefficients of x^2 gives $5=A+C \Rightarrow A=2$. So

$$\int \frac{5x^2+3x-2}{x^3+2x^2}\,dx = \int\left(\frac{2}{x} - \frac{1}{x^2} + \frac{3}{x+2}\right)dx = 2\ln|x| + \frac{1}{x} + 3\ln|x+2| + C.$$

22. $\dfrac{1}{s^2(s-1)^2} = \dfrac{A}{s} + \dfrac{B}{s^2} + \dfrac{C}{s-1} + \dfrac{D}{(s-1)^2} \Rightarrow 1 = As(s-1)^2 + B(s-1)^2 + Cs^2(s-1) + Ds^2$. Set $s=0$, giving $B=1$. Then set $s=1$ to get $D=1$. Equate the coefficients of s^3 to get $0=A+C$ or $A=-C$, and finally set $s=2$ to get $1=2A+1-4A+4$ or $A=2$. Now

$$\int \frac{ds}{s^2(s-1)^2} = \int\left[\frac{2}{s} + \frac{1}{s^2} - \frac{2}{s-1} + \frac{1}{(s-1)^2}\right]ds = 2\ln|s| - \frac{1}{s} - 2\ln|s-1| - \frac{1}{s-1} + C.$$

23. $\dfrac{x^2}{(x+1)^3} = \dfrac{A}{x+1} + \dfrac{B}{(x+1)^2} + \dfrac{C}{(x+1)^3}$. Multiply by $(x+1)^3$ to get $x^2 = A(x+1)^2 + B(x+1) + C$. Setting $x=-1$ gives $C=1$. Equating the coefficients of x^2 gives $A=1$, and setting $x=0$ gives $B=-2$.

Now $\displaystyle\int \frac{x^2\,dx}{(x+1)^3} = \int\left[\frac{1}{x+1} - \frac{2}{(x+1)^2} + \frac{1}{(x+1)^3}\right]dx = \ln|x+1| + \frac{2}{x+1} - \frac{1}{2(x+1)^2} + C.$

24. $\dfrac{x}{x+1} = \dfrac{(x+1)-1}{x+1} = 1 - \dfrac{1}{x+1}$, so $\dfrac{x^3}{(x+1)^3} = \left[1-\dfrac{1}{x+1}\right]^3 = 1 - \dfrac{3}{x+1} + \dfrac{3}{(x+1)^2} - \dfrac{1}{(x+1)^3}$. Thus,

$$\int \frac{x^3}{(x+1)^3}\,dx = \int\left[1 - \frac{3}{x+1} + \frac{3}{(x+1)^2} - \frac{1}{(x+1)^3}\right]dx = x - 3\ln|x+1| - \frac{3}{x+1} + \frac{1}{2(x+1)^2} + C.$$

25. $\dfrac{10}{(x-1)(x^2+9)} = \dfrac{A}{x-1} + \dfrac{Bx+C}{x^2+9}$. Multiply both sides by $(x-1)(x^2+9)$ to get $10 = A(x^2+9) + (Bx+C)(x-1)$ $(*)$. Substituting 1 for x gives $10 = 10A \;\Leftrightarrow\; A = 1$. Substituting 0 for x gives $10 = 9A - C \;\Rightarrow\; C = 9(1) - 10 = -1$. The coefficients of the x^2-terms in $(*)$ must be equal, so $0 = A + B \;\Rightarrow\; B = -1$. Thus,

$$\int \frac{10}{(x-1)(x^2+9)}\,dx = \int\left(\frac{1}{x-1} + \frac{-x-1}{x^2+9}\right)dx = \int\left(\frac{1}{x-1} - \frac{x}{x^2+9} - \frac{1}{x^2+9}\right)dx$$
$$= \ln|x-1| - \tfrac{1}{2}\ln(x^2+9) \;[\text{let } u = x^2+9] - \tfrac{1}{3}\tan^{-1}\left(\tfrac{x}{3}\right) \;[\text{Formula 10}] + C$$

26. $\dfrac{x^2-x+6}{x^3+3x} = \dfrac{x^2-x+6}{x(x^2+3)} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+3}$. Multiply by $x(x^2+3)$ to get $x^2 - x + 6 = A(x^2+3) + (Bx+C)x$. Substituting 0 for x gives $6 = 3A \;\Leftrightarrow\; A = 2$. The coefficients of the x^2-terms must be equal, so $1 = A + B \;\Rightarrow\; B = 1 - 2 = -1$. The coefficients of the x-terms must be equal, so $-1 = C$. Thus,

$$\int \frac{x^2-x+6}{x^3+3x}\,dx = \int\left(\frac{2}{x} + \frac{-x-1}{x^2+3}\right)dx = \int\left(\frac{2}{x} - \frac{x}{x^2+3} - \frac{1}{x^2+3}\right)dx$$
$$= 2\ln|x| - \frac{1}{2}\ln(x^2+3) - \frac{1}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}} + C$$

27. $\dfrac{x^3+x^2+2x+1}{(x^2+1)(x^2+2)} = \dfrac{Ax+B}{x^2+1} + \dfrac{Cx+D}{x^2+2}$. Multiply both sides by $(x^2+1)(x^2+2)$ to get

$x^3 + x^2 + 2x + 1 = (Ax+B)(x^2+2) + (Cx+D)(x^2+1) \;\Leftrightarrow$

$x^3 + x^2 + 2x + 1 = (Ax^3 + Bx^2 + 2Ax + 2B) + (Cx^3 + Dx^2 + Cx + D) \;\Leftrightarrow$

$x^3 + x^2 + 2x + 1 = (A+C)x^3 + (B+D)x^2 + (2A+C)x + (2B+D)$. Comparing coefficients gives us the following system of equations:

$$A + C = 1 \quad \textbf{(1)} \qquad B + D = 1 \quad \textbf{(2)}$$
$$2A + C = 2 \quad \textbf{(3)} \qquad 2B + D = 1 \quad \textbf{(4)}$$

Subtracting equation **(1)** from equation **(3)** gives us $A = 1$, so $C = 0$. Subtracting equation **(2)** from equation **(4)** gives us $B = 0$, so $D = 1$. Thus, $I = \displaystyle\int \frac{x^3+x^2+2x+1}{(x^2+1)(x^2+2)}\,dx = \int\left(\frac{x}{x^2+1} + \frac{1}{x^2+2}\right)dx$. For $\displaystyle\int \frac{x}{x^2+1}\,dx$, let $u = x^2+1$ so $du = 2x\,dx$ and then $\displaystyle\int \frac{x}{x^2+1}\,dx = \frac{1}{2}\int\frac{1}{u}\,du = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln(x^2+1) + C$. For $\displaystyle\int \frac{1}{x^2+2}\,dx$, use Formula 10 with $a = \sqrt{2}$. So $\displaystyle\int \frac{1}{x^2+2}\,dx = \int \frac{1}{x^2+(\sqrt{2})^2}\,dx = \frac{1}{\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}} + C$.

Thus, $I = \dfrac{1}{2}\ln(x^2+1) + \dfrac{1}{\sqrt{2}}\tan^{-1}\dfrac{x}{\sqrt{2}} + C$.

28. $\dfrac{x^2-2x-1}{(x-1)^2(x^2+1)} = \dfrac{A}{x-1} + \dfrac{B}{(x-1)^2} + \dfrac{Cx+D}{x^2+1} \;\Rightarrow$

$x^2 - 2x - 1 = A(x-1)(x^2+1) + B(x^2+1) + (Cx+D)(x-1)^2$. Setting $x = 1$ gives $B = -1$. Equating the coefficients of x^3 gives $A = -C$. Equating the constant terms gives $-1 = -A - 1 + D$, so $D = A$, and setting $x = 2$ gives $-1 = 5A - 5 - 2A + A$ or $A = 1$. We have

$$\int \frac{x^2 - 2x - 1}{(x-1)^2(x^2+1)}\,dx = \int \left[\frac{1}{x-1} - \frac{1}{(x-1)^2} - \frac{x-1}{x^2+1}\right] dx$$

$$= \ln|x-1| + \frac{1}{x-1} - \tfrac{1}{2}\ln(x^2+1) + \tan^{-1}x + C$$

29. $$\int \frac{x+4}{x^2+2x+5}\,dx = \int \frac{x+1}{x^2+2x+5}\,dx + \int \frac{3}{x^2+2x+5}\,dx = \frac{1}{2}\int \frac{(2x+2)\,dx}{x^2+2x+5} + \int \frac{3\,dx}{(x+1)^2+4}$$

$$= \frac{1}{2}\ln|x^2+2x+5| + 3\int \frac{2\,du}{4(u^2+1)} \qquad \left[\begin{array}{l}\text{where } x+1 = 2u,\\ \text{and } dx = 2\,du\end{array}\right]$$

$$= \frac{1}{2}\ln(x^2+2x+5) + \frac{3}{2}\tan^{-1}u + C = \frac{1}{2}\ln(x^2+2x+5) + \frac{3}{2}\tan^{-1}\left(\frac{x+1}{2}\right) + C$$

30. $\dfrac{x^3 - 2x^2 + x + 1}{x^4 + 5x^2 + 4} = \dfrac{x^3 - 2x^2 + x + 1}{(x^2+1)(x^2+4)} = \dfrac{Ax+B}{x^2+1} + \dfrac{Cx+D}{x^2+4} \Rightarrow$

$x^3 - 2x^2 + x + 1 = (Ax+B)(x^2+4) + (Cx+D)(x^2+1)$. Equating coefficients gives $A + C = 1$, $B + D = -2$, $4A + C = 1$, $4B + D = 1 \Rightarrow A = 0, C = 1, B = 1, D = -3$. Now

$$\int \frac{x^3 - 2x^2 + x + 1}{x^4 + 5x^2 + 4}\,dx = \int \frac{dx}{x^2+1} + \int \frac{x-3}{x^2+4}\,dx = \tan^{-1}x + \tfrac{1}{2}\ln(x^2+4) - \tfrac{3}{2}\tan^{-1}(x/2) + C.$$

31. $\dfrac{1}{x^3-1} = \dfrac{1}{(x-1)(x^2+x+1)} = \dfrac{A}{x-1} + \dfrac{Bx+C}{x^2+x+1} \Rightarrow 1 = A(x^2+x+1) + (Bx+C)(x-1)$.

Take $x = 1$ to get $A = \frac{1}{3}$. Equating coefficients of x^2 and then comparing the constant terms, we get $0 = \frac{1}{3} + B$, $1 = \frac{1}{3} - C$, so $B = -\frac{1}{3}$, $C = -\frac{2}{3} \Rightarrow$

$$\int \frac{1}{x^3-1}\,dx = \int \frac{\frac{1}{3}}{x-1}\,dx + \int \frac{-\frac{1}{3}x - \frac{2}{3}}{x^2+x+1}\,dx = \tfrac{1}{3}\ln|x-1| - \frac{1}{3}\int \frac{x+2}{x^2+x+1}\,dx$$

$$= \tfrac{1}{3}\ln|x-1| - \frac{1}{3}\int \frac{x+1/2}{x^2+x+1}\,dx - \frac{1}{3}\int \frac{(3/2)\,dx}{(x+1/2)^2 + 3/4}$$

$$= \tfrac{1}{3}\ln|x-1| - \tfrac{1}{6}\ln(x^2+x+1) - \tfrac{1}{2}\left(\tfrac{2}{\sqrt{3}}\right)\tan^{-1}\left(\frac{x+\frac{1}{2}}{\sqrt{3}/2}\right) + K$$

$$= \tfrac{1}{3}\ln|x-1| - \tfrac{1}{6}\ln(x^2+x+1) - \tfrac{1}{\sqrt{3}}\tan^{-1}\left(\tfrac{1}{\sqrt{3}}(2x+1)\right) + K$$

32. $$\int_0^1 \frac{x}{x^2+4x+13}\,dx = \int_0^1 \frac{\frac{1}{2}(2x+4)}{x^2+4x+13}\,dx - 2\int_0^1 \frac{dx}{(x+2)^2+9}$$

$$= \frac{1}{2}\int_{13}^{18} \frac{dy}{y} - 2\int_{2/3}^{1} \frac{3\,du}{9u^2+9} \qquad \left[\begin{array}{l}\text{where } y = x^2+4x+13,\ dy = (2x+4)\,dx,\\ x+2 = 3u, \text{ and } dx = 3\,du\end{array}\right]$$

$$= \tfrac{1}{2}\left[\ln y\right]_{13}^{18} - \tfrac{2}{3}\left[\tan^{-1}u\right]_{2/3}^{1} = \tfrac{1}{2}\ln\tfrac{18}{13} - \tfrac{2}{3}\left(\tfrac{\pi}{4} - \tan^{-1}\left(\tfrac{2}{3}\right)\right)$$

$$= \tfrac{1}{2}\ln\tfrac{18}{13} - \tfrac{\pi}{6} + \tfrac{2}{3}\tan^{-1}\left(\tfrac{2}{3}\right)$$

33. Let $u = x^3 + 3x^2 + 4$. Then $du = 3(x^2 + 2x)\,dx \Rightarrow$

$$\int_2^5 \frac{x^2 + 2x}{x^3 + 3x^2 + 4}\,dx = \frac{1}{3}\int_{24}^{204} \frac{du}{u} = \tfrac{1}{3}[\ln u]_{24}^{204} = \tfrac{1}{3}(\ln 204 - \ln 24) = \tfrac{1}{3}\ln\tfrac{204}{24} = \tfrac{1}{3}\ln\tfrac{17}{2}.$$

34. $$\frac{x^3}{x^3+1} = \frac{(x^3+1)-1}{x^3+1} = 1 - \frac{1}{x^3+1} = 1 - \left(\frac{A}{x+1} + \frac{Bx+C}{x^2-x+1}\right) \Rightarrow$$

$1 = A(x^2 - x + 1) + (Bx + C)(x + 1)$. Equate the terms of degree 2, 1 and 0 to get $0 = A + B$, $0 = -A + B + C$, $1 = A + C$. Solve the three equations to get $A = \frac{1}{3}$, $B = -\frac{1}{3}$, and $C = \frac{2}{3}$. So

$$\int \frac{x^3}{x^3+1}\,dx = \int \left[1 - \frac{\frac{1}{3}}{x+1} + \frac{\frac{1}{3}x - \frac{2}{3}}{x^2 - x + 1}\right] dx$$

$$= x - \tfrac{1}{3}\ln|x+1| + \frac{1}{6}\int \frac{2x-1}{x^2-x+1}\,dx - \frac{1}{2}\int \frac{dx}{\left(x-\frac{1}{2}\right)^2 + \frac{3}{4}}$$

$$= x - \tfrac{1}{3}\ln|x+1| + \tfrac{1}{6}\ln(x^2 - x + 1) - \tfrac{1}{\sqrt{3}}\tan^{-1}\!\left(\tfrac{1}{\sqrt{3}}(2x-1)\right) + K$$

35. $$\frac{1}{x^4 - x^2} = \frac{1}{x^2(x-1)(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{D}{x+1}.$$ Multiply by $x^2(x-1)(x+1)$ to get

$1 = Ax(x-1)(x+1) + B(x-1)(x+1) + Cx^2(x+1) + Dx^2(x-1)$. Setting $x = 1$ gives $C = \frac{1}{2}$, taking $x = -1$ gives $D = -\frac{1}{2}$. Equating the coefficients of x^3 gives $0 = A + C + D = A$. Finally, setting $x = 0$ yields $B = -1$. Now $$\int \frac{dx}{x^4 - x^2} = \int \left[\frac{-1}{x^2} + \frac{1/2}{x-1} - \frac{1/2}{x+1}\right] dx = \frac{1}{x} + \tfrac{1}{2}\ln\left|\frac{x-1}{x+1}\right| + C.$$

36. Let $u = x^4 + 5x^2 + 4 \Rightarrow du = (4x^3 + 10x)\,dx = 2(2x^3 + 5x)\,dx$, so

$$\int_0^1 \frac{2x^3 + 5x}{x^4 + 5x^2 + 4}\,dx = \frac{1}{2}\int_4^{10} \frac{du}{u} = \tfrac{1}{2}\Big[\ln|u|\Big]_4^{10} = \tfrac{1}{2}(\ln 10 - \ln 4) = \tfrac{1}{2}\ln\tfrac{5}{2}.$$

37. $$\int \frac{x-3}{(x^2+2x+4)^2}\,dx = \int \frac{x-3}{\left[(x+1)^2+3\right]^2}\,dx = \int \frac{u-4}{(u^2+3)^2}\,du \qquad [\text{with } u = x + 1]$$

$$= \int \frac{u\,du}{(u^2+3)^2} - 4\int \frac{du}{(u^2+3)^2} = \frac{1}{2}\int \frac{dv}{v^2} - 4\int \frac{\sqrt{3}\sec^2\theta\,d\theta}{9\sec^4\theta} \qquad \left[\begin{array}{c} v = u^2 + 3 \text{ in the first integral;} \\ u = \sqrt{3}\tan\theta \text{ in the second} \end{array}\right]$$

$$= \frac{-1}{(2v)} - \frac{4\sqrt{3}}{9}\int \cos^2\theta\,d\theta = \frac{-1}{2(u^2+3)} - \frac{2\sqrt{3}}{9}(\theta + \sin\theta\cos\theta) + C$$

$$= \frac{-1}{2(x^2+2x+4)} - \frac{2\sqrt{3}}{9}\left[\tan^{-1}\left(\frac{x+1}{\sqrt{3}}\right) + \frac{\sqrt{3}(x+1)}{x^2+2x+4}\right] + C$$

$$= \frac{-1}{2(x^2+2x+4)} - \frac{2\sqrt{3}}{9}\tan^{-1}\left(\frac{x+1}{\sqrt{3}}\right) - \frac{2(x+1)}{3(x^2+2x+4)} + C$$

38. $$\frac{x^4+1}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2} \Rightarrow x^4 + 1 = A(x^2+1)^2 + (Bx+C)x(x^2+1) + (Dx+E)x.$$

Setting $x = 0$ gives $A = 1$, and equating the coefficients of x^4 gives $1 = A + B$, so $B = 0$. Now

$$\frac{C}{x^2+1}+\frac{Dx+E}{(x^2+1)^2}=\frac{x^4+1}{x(x^2+1)^2}-\frac{1}{x}=\frac{1}{x}\left[\frac{x^4+1-(x^4+2x^2+1)}{(x^2+1)^2}\right]=\frac{-2x}{(x^2+1)^2}$$, so we can take $C=0$,

$D=-2$, and $E=0$. Hence, $\displaystyle\int\frac{x^4+1}{x(x^2+1)^2}dx=\int\left[\frac{1}{x}-\frac{2x}{(x^2+1)^2}\right]dx=\ln|x|+\frac{1}{x^2+1}+C.$

39. Let $u=\sqrt{x+1}$. Then $x=u^2-1$, $dx=2u\,du$ $\Rightarrow$

$$\int\frac{dx}{x\sqrt{x+1}}=\int\frac{2u\,du}{(u^2-1)\,u}=2\int\frac{du}{u^2-1}=\ln\left|\frac{u-1}{u+1}\right|+C=\ln\left|\frac{\sqrt{x+1}-1}{\sqrt{x+1}+1}\right|+C.$$

40. Let $u=\sqrt{x+2}$. Then $x=u^2-2$, $dx=2u\,du$ $\Rightarrow$

$$I=\int\frac{dx}{x-\sqrt{x+2}}=\int\frac{2u\,du}{u^2-2-u}=2\int\frac{u\,du}{u^2-u-2}\text{ and }\frac{u}{u^2-u-2}=\frac{A}{u-2}+\frac{B}{u+1}\Rightarrow$$

$u=A(u+1)+B(u-2)$. Substituting -1 for u gives $-1=-3B$ $\Leftrightarrow$ $B=\frac{1}{3}$ and substituting 2 for u gives $2=3A$ $\Leftrightarrow$ $A=\frac{2}{3}$. Thus,

$$\begin{aligned}I&=\frac{2}{3}\int\left[\frac{2}{u-2}+\frac{1}{u+1}\right]du=\tfrac{2}{3}(2\ln|u-2|+\ln|u+1|)+C\\&=\tfrac{2}{3}\left[2\ln\left|\sqrt{x+2}-2\right|+\ln\left(\sqrt{x+2}+1\right)\right]+C\end{aligned}$$

41. Let $u=\sqrt{x}$, so $u^2=x$ and $dx=2u\,du$. Thus,

$$\begin{aligned}\int_9^{16}\frac{\sqrt{x}}{x-4}\,dx&=\int_3^4\frac{u}{u^2-4}2u\,du=2\int_3^4\frac{u^2}{u^2-4}\,du=2\int_3^4\left(1+\frac{4}{u^2-4}\right)du\quad\text{[by long division]}\\&=2+8\int_3^4\frac{du}{(u+2)(u-2)}.\quad(*)\end{aligned}$$

Multiply $\dfrac{1}{(u+2)(u-2)}=\dfrac{A}{u+2}+\dfrac{B}{u-2}$ by $(u+2)(u-2)$ to get $1=A(u-2)+B(u+2)$. Equating coefficients we get $A+B=0$ and $-2A+2B=1$. Solving gives us $B=\frac{1}{4}$ and $A=-\frac{1}{4}$, so

$\dfrac{1}{(u+2)(u-2)}=\dfrac{-1/4}{u+2}+\dfrac{1/4}{u-2}$ and $(*)$ is

$$\begin{aligned}2+8\int_3^4\left(\frac{-1/4}{u+2}+\frac{1/4}{u-2}\right)du&=2+8\left[-\tfrac{1}{4}\ln|u+2|+\tfrac{1}{4}\ln|u-2|\right]_3^4\\&=2+\Big[2\ln|u-2|-2\ln|u+2|\Big]_3^4=2+2\left[\ln\left|\frac{u-2}{u+2}\right|\right]_3^4\\&=2+2\left(\ln\tfrac{2}{6}-\ln\tfrac{1}{5}\right)=2+2\ln\tfrac{2/6}{1/5}\\&=2+2\ln\tfrac{5}{3}\ \text{ or }\ 2+\ln\left(\tfrac{5}{3}\right)^2=2+\ln\tfrac{25}{9}\end{aligned}$$

42. Let $u=\sqrt[3]{x}$. Then $x=u^3$, $dx=3u^2\,du$ $\Rightarrow$

$$\begin{aligned}\int_0^1\frac{1}{1+\sqrt[3]{x}}\,dx&=\int_0^1\frac{3u^2\,du}{1+u}=\int_0^1\left(3u-3+\frac{3}{1+u}\right)du=\left[\tfrac{3}{2}u^2-3u+3\ln(1+u)\right]_0^1\\&=3\left(\ln 2-\tfrac{1}{2}\right)\end{aligned}$$

43. Let $u = \sqrt[3]{x^2+1}$. Then $x^2 = u^3 - 1$, $2x\,dx = 3u^2\,du \Rightarrow$

$$\int \frac{x^3\,dx}{\sqrt[3]{x^2+1}} = \int \frac{(u^3-1)\frac{3}{2}u^2\,du}{u} = \frac{3}{2}\int (u^4-u)\,du = \frac{3}{10}u^5 - \frac{3}{4}u^2 + C$$
$$= \frac{3}{10}(x^2+1)^{5/3} - \frac{3}{4}(x^2+1)^{2/3} + C$$

44. Let $u = \sqrt{x}$. Then $x = u^2$, $dx = 2u\,du \Rightarrow$

$$\int_{1/3}^{3} \frac{\sqrt{x}}{x^2+x}\,dx = \int_{1/\sqrt{3}}^{\sqrt{3}} \frac{u\cdot 2u\,du}{u^4+u^2} = 2\int_{1/\sqrt{3}}^{\sqrt{3}} \frac{du}{u^2+1} = 2\big[\tan^{-1} u\big]_{1/\sqrt{3}}^{\sqrt{3}} = 2\left(\frac{\pi}{3} - \frac{\pi}{6}\right) = \frac{\pi}{3}$$

45. If we were to substitute $u = \sqrt{x}$, then the square root would disappear but a cube root would remain. On the other hand, the substitution $u = \sqrt[3]{x}$ would eliminate the cube root but leave a square root. We can eliminate both roots by means of the substitution $u = \sqrt[6]{x}$. (Note that 6 is the least common multiple of 2 and 3.)

Let $u = \sqrt[6]{x}$. Then $x = u^6$, so $dx = 6u^5\,du$ and $\sqrt{x} = u^3$, $\sqrt[3]{x} = u^2$. Thus,

$$\int \frac{dx}{\sqrt{x} - \sqrt[3]{x}} = \int \frac{6u^5\,du}{u^3-u^2} = 6\int \frac{u^5}{u^2(u-1)}\,du = 6\int \frac{u^3}{u-1}\,du$$
$$= 6\int \left(u^2+u+1+\frac{1}{u-1}\right)du \quad \text{[by long division]}$$
$$= 6\left(\frac{1}{3}u^3 + \frac{1}{2}u^2 + u + \ln|u-1|\right) + C = 2\sqrt{x} + 3\sqrt[3]{x} + 6\sqrt[6]{x} + 6\ln\left|\sqrt[6]{x} - 1\right| + C$$

46. Let $u = \sqrt[12]{x}$. Then $x = u^{12}$, $dx = 12u^{11}\,du \Rightarrow$

$$\int \frac{dx}{\sqrt[3]{x} + \sqrt[4]{x}} = \int \frac{12u^{11}du}{u^4+u^3} = 12\int \frac{u^8du}{u+1} = 12\int \left(u^7 - u^6 + u^5 - u^4 + u^3 - u^2 + u - 1 + \frac{1}{u+1}\right)du$$
$$= \frac{3}{2}u^8 - \frac{12}{7}u^7 + 2u^6 - \frac{12}{5}u^5 + 3u^4 - 4u^3 + 6u^2 - 12u + 12\ln|u+1| + C$$
$$= \frac{3}{2}x^{2/3} - \frac{12}{7}x^{7/12} + 2\sqrt{x} - \frac{12}{5}x^{5/12} + 3\sqrt[3]{x} - 4\sqrt[4]{x} + 6\sqrt[6]{x} - 12\sqrt[12]{x} + 12\ln\left(\sqrt[12]{x}+1\right) + C$$

47. Let $u = e^x$. Then $x = \ln u$, $dx = \dfrac{du}{u} \Rightarrow$

$$\int \frac{e^{2x}\,dx}{e^{2x}+3e^x+2} = \int \frac{u^2\,(du/u)}{u^2+3u+2} = \int \frac{u\,du}{(u+1)(u+2)} = \int \left[\frac{-1}{u+1} + \frac{2}{u+2}\right]du$$
$$= 2\ln|u+2| - \ln|u+1| + C = \ln\left[(e^x+2)^2/(e^x+1)\right] + C$$

48. Let $u = \sin x$. Then $du = \cos x\,dx \Rightarrow$

$$\int \frac{\cos x\,dx}{\sin^2 x + \sin x} = \int \frac{du}{u^2+u} = \int \frac{du}{u(u+1)} = \int \left[\frac{1}{u} - \frac{1}{u+1}\right]du$$
$$= \ln\left|\frac{u}{u+1}\right| + C = \ln\left|\frac{\sin x}{1+\sin x}\right| + C$$

49. Let $u = \ln(x^2 - x + 2)$, $dv = dx$. Then $du = \dfrac{2x - 1}{x^2 - x + 2}\,dx$, $v = x$, and (by integration by parts)

$$\int \ln(x^2 - x + 2)\,dx = x\ln(x^2 - x + 2) - \int \frac{2x^2 - x}{x^2 - x + 2}\,dx = x\ln(x^2 - x + 2) - \int\left(2 + \frac{x - 4}{x^2 - x + 2}\right)dx$$

$$= x\ln(x^2 - x + 2) - 2x - \int \frac{\frac{1}{2}(2x - 1)}{x^2 - x + 2}\,dx + \frac{7}{2}\int \frac{dx}{(x - \frac{1}{2})^2 + \frac{7}{4}}$$

$$= x\ln(x^2 - x + 2) - 2x - \frac{1}{2}\ln(x^2 - x + 2) + \frac{7}{2}\int \frac{\frac{\sqrt{7}}{2}\,du}{\frac{7}{4}(u^2 + 1)} \quad \left[\begin{array}{r} \text{where } x - \frac{1}{2} = \frac{\sqrt{7}}{2}u, \\ dx = \frac{\sqrt{7}}{2}\,du, \\ (x - \frac{1}{2})^2 + \frac{7}{4} = \frac{7}{4}(u^2 + 1) \end{array}\right]$$

$$= (x - \tfrac{1}{2})\ln(x^2 - x + 2) - 2x + \sqrt{7}\tan^{-1}u + C$$

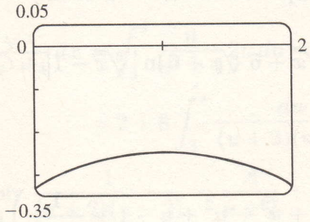
$$= (x - \tfrac{1}{2})\ln(x^2 - x + 2) - 2x + \sqrt{7}\tan^{-1}\frac{2x - 1}{\sqrt{7}} + C$$

50. Let $u = \tan^{-1}x$, $dv = x\,dx \Rightarrow du = dx/(1 + x^2)$, $v = \frac{1}{2}x^2$.

Then $\int x\tan^{-1}x\,dx = \frac{1}{2}x^2\tan^{-1}x - \dfrac{1}{2}\displaystyle\int \frac{x^2}{1 + x^2}\,dx$. To evaluate the last integral, use long division or observe that $\displaystyle\int \frac{x^2}{1 + x^2}\,dx = \int \frac{(1 + x^2) - 1}{1 + x^2}\,dx = \int 1\,dx - \int \frac{1}{1 + x^2}\,dx = x - \tan^{-1}x + C_1$. So

$\int x\tan^{-1}x\,dx = \frac{1}{2}x^2\tan^{-1}x - \frac{1}{2}(x - \tan^{-1}x + C_1) = \frac{1}{2}(x^2\tan^{-1}x + \tan^{-1}x - x) + C$.

51.



From the graph, we see that the integral will be negative, and we guess that the area is about the same as that of a rectangle with width 2 and height 0.3, so we estimate the integral to be $-(2\cdot 0.3) = -0.6$. Now

$$\frac{1}{x^2 - 2x - 3} = \frac{1}{(x - 3)(x + 1)} = \frac{A}{x - 3} + \frac{B}{x + 1} \Leftrightarrow$$

$1 = (A + B)x + A - 3B$, so $A = -B$ and $A - 3B = 1 \Leftrightarrow A = \frac{1}{4}$ and $B = -\frac{1}{4}$, so the integral becomes

$$\int_0^2 \frac{dx}{x^2 - 2x - 3} = \frac{1}{4}\int_0^2 \frac{dx}{x - 3} - \frac{1}{4}\int_0^2 \frac{dx}{x + 1} = \tfrac{1}{4}\Big[\ln|x - 3| - \ln|x + 1|\Big]_0^2$$

$$= \frac{1}{4}\left[\ln\left|\frac{x - 3}{x + 1}\right|\right]_0^2 = \tfrac{1}{4}(\ln\tfrac{1}{3} - \ln 3) = -\tfrac{1}{2}\ln 3 \approx -0.55$$

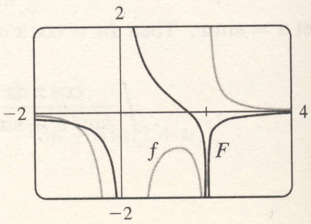
52. $\dfrac{1}{x^3 - 2x^2} = \dfrac{1}{x^2(x - 2)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x - 2} \Rightarrow 1 = (A + C)x^2 + (B - 2A)x - 2B$, so $A + C = B - 2A = 0$

and $-2B = 1 \Rightarrow B = -\frac{1}{2}$, $A = -\frac{1}{4}$, and $C = \frac{1}{4}$. So the general antiderivative of $\dfrac{1}{x^3 - 2x^2}$ is

$$\int \frac{dx}{x^3 - 2x^2} = -\frac{1}{4}\int \frac{dx}{x} - \frac{1}{2}\int \frac{dx}{x^2} + \frac{1}{4}\int \frac{dx}{x - 2}$$

$$= -\tfrac{1}{4}\ln|x| - \tfrac{1}{2}(-1/x) + \tfrac{1}{4}\ln|x - 2| + C$$

$$= \frac{1}{4}\ln\left|\frac{x - 2}{x}\right| + \frac{1}{2x} + C$$



We plot this function with $C = 0$ on the same screen as $y = \dfrac{1}{x^3 - 2x^2}$.