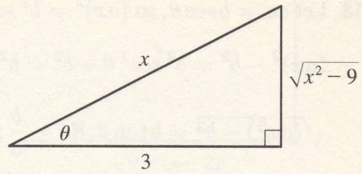
13. Let $x = 3\sec\theta$, where $0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$. Then

$dx = 3\sec\theta\tan\theta\,d\theta$ and $\sqrt{x^2-9} = 3\tan\theta$, so

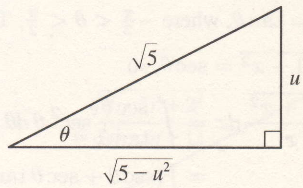


$$\int \frac{\sqrt{x^2-9}}{x^3}\,dx = \int \frac{3\tan\theta}{27\sec^3\theta}\,3\sec\theta\tan\theta\,d\theta = \frac{1}{3}\int \frac{\tan^2\theta}{\sec^2\theta}\,d\theta$$

$$= \tfrac{1}{3}\int \sin^2\theta\,d\theta = \tfrac{1}{3}\int \tfrac{1}{2}(1-\cos 2\theta)\,d\theta = \tfrac{1}{6}\theta - \tfrac{1}{12}\sin 2\theta + C = \tfrac{1}{6}\theta - \tfrac{1}{6}\sin\theta\cos\theta + C$$

$$= \frac{1}{6}\sec^{-1}\!\left(\frac{x}{3}\right) - \frac{1}{6}\frac{\sqrt{x^2-9}}{x}\frac{3}{x} + C = \frac{1}{6}\sec^{-1}\!\left(\frac{x}{3}\right) - \frac{\sqrt{x^2-9}}{2x^2} + C$$

14. Let $u = \sqrt{5}\sin\theta$, so $du = \sqrt{5}\cos\theta\,d\theta$. Then



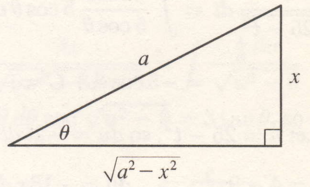
$$\int \frac{du}{u\sqrt{5-u^2}} = \int \frac{1}{\sqrt{5}\sin\theta\cdot\sqrt{5}\cos\theta}\sqrt{5}\cos\theta\,d\theta = \frac{1}{\sqrt{5}}\int \csc\theta\,d\theta$$

$$= \tfrac{1}{\sqrt{5}}\ln|\csc\theta - \cot\theta| + C \quad \text{[by Exercise 8.2.39]}$$

$$= \frac{1}{\sqrt{5}}\ln\left|\frac{\sqrt{5}}{u} - \frac{\sqrt{5-u^2}}{u}\right| + C$$

$$= \frac{1}{\sqrt{5}}\ln\left|\frac{\sqrt{5}-\sqrt{5-u^2}}{u}\right| + C$$

15. Let $x = a\sin\theta$, where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. Then $dx = a\cos\theta\,d\theta$ and



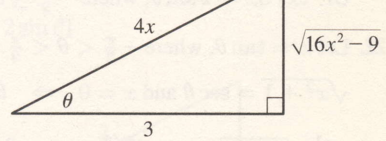
$$\int \frac{x^2\,dx}{(a^2-x^2)^{3/2}} = \int \frac{a^2\sin^2\theta\,a\cos\theta\,d\theta}{a^3\cos^3\theta} = \int \tan^2\theta\,d\theta$$

$$= \int (\sec^2\theta - 1)\,d\theta = \tan\theta - \theta + C$$

$$= \frac{x}{\sqrt{a^2-x^2}} - \sin^{-1}\frac{x}{a} + C$$

16. Let $4x = 3\sec\theta$, where $0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$. Then

$dx = \frac{3}{4}\sec\theta\tan\theta\,d\theta$ and $\sqrt{16x^2-9} = 3\tan\theta$, so

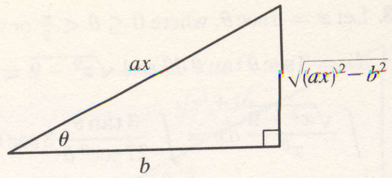


$$\int \frac{dx}{x^2\sqrt{16x^2-9}} = \int \frac{\frac{3}{4}\sec\theta\tan\theta\,d\theta}{\left(\frac{3}{4}\right)^2\sec^2\theta\,3\tan\theta}$$

$$= \frac{4}{9}\int \cos\theta\,d\theta = \frac{4}{9}\sin\theta + C = \frac{4}{9}\frac{\sqrt{16x^2-9}}{4x} + C = \frac{\sqrt{16x^2-9}}{9x} + C$$

17. Let $u = x^2 - 7$, so $du = 2x\,dx$. Then $\displaystyle\int \frac{x}{\sqrt{x^2-7}}\,dx = \frac{1}{2}\int \frac{1}{\sqrt{u}}\,du = \tfrac{1}{2}\cdot 2\sqrt{u} + C = \sqrt{x^2-7} + C.$

18. Let $ax = b\sec\theta$, so $(ax)^2 = b^2\sec^2\theta \;\Rightarrow$



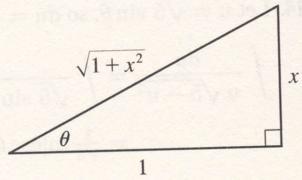
$(ax)^2 - b^2 = b^2\sec^2\theta - b^2 = b^2(\sec^2\theta - 1) = b^2\tan^2\theta$. So

$\sqrt{(ax^2) - b^2} = b\tan\theta$, $dx = \dfrac{b}{a}\sec\theta\tan\theta\,d\theta$, and

$$\int \frac{dx}{\left[(ax)^2 - b^2\right]^{3/2}} = \int \frac{\frac{b}{a}\sec\theta\tan\theta}{b^3\tan^3\theta}\,d\theta = \frac{1}{ab^2}\int\frac{\sec\theta}{\tan^2\theta}\,d\theta = \frac{1}{ab^2}\int\frac{\cos\theta}{\sin^2\theta}\,d\theta = \frac{1}{ab^2}\int\csc\theta\cot\theta\,d\theta$$

$$= -\frac{1}{ab^2}\csc\theta + C = -\frac{1}{ab^2}\frac{ax}{\sqrt{(ax)^2 - b^2}} + C = -\frac{x}{b^2\sqrt{(ax)^2 - b^2}} + C$$

19. Let $x = \tan\theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Then $dx = \sec^2\theta\,d\theta$



and $\sqrt{1 + x^2} = \sec\theta$, so

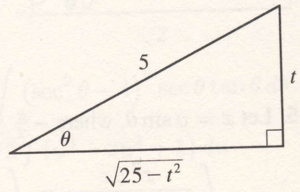
$$\int \frac{\sqrt{1+x^2}}{x}\,dx = \int \frac{\sec\theta}{\tan\theta}\sec^2\theta\,d\theta = \int\frac{\sec\theta}{\tan\theta}(1 + \tan^2\theta)\,d\theta$$

$$= \int(\csc\theta + \sec\theta\tan\theta)\,d\theta$$

$$= \ln|\csc\theta - \cot\theta| + \sec\theta + C \quad \text{[by Exercise 8.2.39]}$$

$$= \ln\left|\frac{\sqrt{1+x^2}}{x} - \frac{1}{x}\right| + \frac{\sqrt{1+x^2}}{1} + C = \ln\left|\frac{\sqrt{1+x^2} - 1}{x}\right| + \sqrt{1+x^2} + C$$

20. Let $t = 5\sin\theta$, where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. Then $dt = 5\cos\theta\,d\theta$



and $\sqrt{25 - t^2} = 5\cos\theta$, so

$$\int\frac{t}{\sqrt{25 - t^2}}\,dt = \int\frac{5\sin\theta}{5\cos\theta}5\cos\theta\,d\theta = 5\int\sin\theta\,d\theta$$

$$= -5\cos\theta + C = -5\cdot\frac{\sqrt{25 - t^2}}{5} + C = -\sqrt{25 - t^2} + C$$

Or: Let $u = 25 - t^2$, so $du = -2t\,dt$.

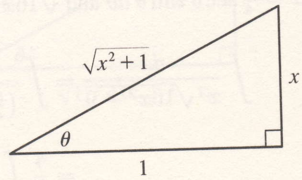
21. Let $u = 4 - 9x^2 \;\Rightarrow\; du = -18x\,dx$. Then $x^2 = \frac{1}{9}(4 - u)$ and

$$\int_0^{2/3} x^3\sqrt{4 - 9x^2}\,dx = \int_4^0 \tfrac{1}{9}(4 - u)u^{1/2}\left(-\tfrac{1}{18}\right)du = \tfrac{1}{162}\int_0^4\left(4u^{1/2} - u^{3/2}\right)du$$

$$= \tfrac{1}{162}\left[\tfrac{8}{3}u^{3/2} - \tfrac{2}{5}u^{5/2}\right]_0^4 = \tfrac{1}{162}\left[\tfrac{64}{3} - \tfrac{64}{5}\right] = \tfrac{64}{1215}$$

Or: Let $3x = 2\sin\theta$, where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.

22. Let $x = \tan\theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Then $dx = \sec^2\theta\,d\theta$,



$\sqrt{x^2 + 1} = \sec\theta$ and $x = 0 \;\Rightarrow\; \theta = 0$, $x = 1 \;\Rightarrow\; \theta = \frac{\pi}{4}$, so

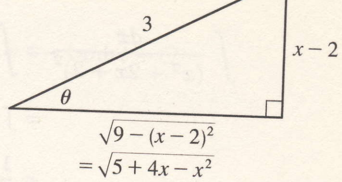
$$\int_0^1\sqrt{x^2 + 1}\,dx = \int_0^{\pi/4}\sec\theta\sec^2\theta\,d\theta = \int_0^{\pi/4}\sec^3\theta\,d\theta$$

$$= \tfrac{1}{2}\Big[\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\Big]_0^{\pi/4} \quad \text{[by Example 8.2.8]}$$

$$= \tfrac{1}{2}\left[\sqrt{2}\cdot 1 + \ln\left(1 + \sqrt{2}\right) - 0 - \ln(1 + 0)\right] = \tfrac{1}{2}\left[\sqrt{2} + \ln\left(1 + \sqrt{2}\right)\right]$$

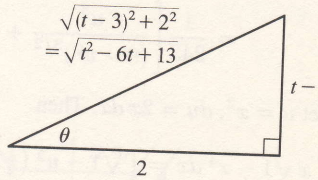
23. $5+4x-x^2=-(x^2-4x+4)+9=-(x-2)^2+9$. Let $x-2=3\sin\theta$, $-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$, so $dx=3\cos\theta\,d\theta$. Then

$$\begin{aligned}\int\sqrt{5+4x-x^2}\,dx&=\int\sqrt{9-(x-2)^2}\,dx=\int\sqrt{9-9\sin^2\theta}\,3\cos\theta\,d\theta\\&=\int\sqrt{9\cos^2\theta}\,3\cos\theta\,d\theta=\int 9\cos^2\theta\,d\theta\\&=\tfrac{9}{2}\int(1+\cos2\theta)\,d\theta=\tfrac{9}{2}\left(\theta+\tfrac{1}{2}\sin2\theta\right)+C\\&=\tfrac{9}{2}\theta+\tfrac{9}{4}\sin2\theta+C=\tfrac{9}{2}\theta+\tfrac{9}{4}(2\sin\theta\cos\theta)+C\\&=\frac{9}{2}\sin^{-1}\left(\frac{x-2}{3}\right)+\frac{9}{2}\cdot\frac{x-2}{3}\cdot\frac{\sqrt{5+4x-x^2}}{3}+C\\&=\frac{9}{2}\sin^{-1}\left(\frac{x-2}{3}\right)+\frac{1}{2}(x-2)\sqrt{5+4x-x^2}+C\end{aligned}$$



24. $t^2-6t+13=(t^2-6t+9)+4=(t-3)^2+2^2$. Let $t-3=2\tan\theta$, so $dt=2\sec^2\theta\,d\theta$. Then

$$\begin{aligned}\int\frac{dt}{\sqrt{t^2-6t+13}}&=\int\frac{1}{\sqrt{(2\tan\theta)^2+2^2}}\,2\sec^2\theta\,d\theta=\int\frac{2\sec^2\theta}{2\sec\theta}\,d\theta\\&=\int\sec\theta\,d\theta=\ln|\sec\theta+\tan\theta|+C_1\quad\text{[by Formula 8.2.1]}\\&=\ln\left|\frac{\sqrt{t^2-6t+13}}{2}+\frac{t-3}{2}\right|+C_1\\&=\ln\left|\sqrt{t^2-6t+13}+t-3\right|+C\qquad\text{where } C=C_1-\ln2\end{aligned}$$

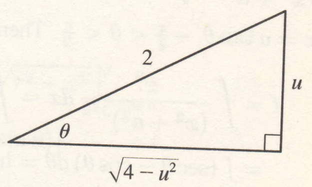


25. $9x^2+6x-8=(3x+1)^2-9$, so let $u=3x+1$, $du=3dx$. Then $\displaystyle\int\frac{dx}{\sqrt{9x^2+6x-8}}=\int\frac{\frac{1}{3}\,du}{\sqrt{u^2-9}}$. Now let $u=3\sec\theta$, where $0\le\theta<\frac{\pi}{2}$ or $\pi\le\theta<\frac{3\pi}{2}$. Then $du=3\sec\theta\tan\theta\,d\theta$ and $\sqrt{u^2-9}=3\tan\theta$, so

$$\begin{aligned}\int\frac{\frac{1}{3}\,du}{\sqrt{u^2-9}}&=\int\frac{\sec\theta\tan\theta\,d\theta}{3\tan\theta}=\tfrac{1}{3}\int\sec\theta d\theta=\tfrac{1}{3}\ln|\sec\theta+\tan\theta|+C_1=\tfrac{1}{3}\ln\left|\frac{u+\sqrt{u^2-9}}{3}\right|+C_1\\&=\tfrac{1}{3}\ln\left|u+\sqrt{u^2-9}\right|+C=\tfrac{1}{3}\ln\left|3x+1+\sqrt{9x^2+6x-8}\right|+C\end{aligned}$$

26. $4x-x^2=-(x^2-4x+4)+4=4-(x-2)^2$, so let $u=x-2$. Then $x=u+2$ and $dx=du$, so

$$\begin{aligned}\int\frac{x^2\,dx}{\sqrt{4x-x^2}}&=\int\frac{(u+2)^2\,du}{\sqrt{4-u^2}}=\int\frac{(2\sin\theta+2)^2}{2\cos\theta}2\cos\theta\,d\theta\quad[\text{Put } u=2\sin\theta]\\&=4\int\left(\sin^2\theta+2\sin\theta+1\right)d\theta\\&=2\int(1-\cos2\theta)\,d\theta+8\int\sin\theta d\theta+4\int d\theta\\&=2\theta-\sin2\theta-8\cos\theta+4\theta+C\\&=6\theta-8\cos\theta-2\sin\theta\cos\theta+C\\&=6\sin^{-1}\left(\tfrac{1}{2}u\right)-4\sqrt{4-u^2}-\tfrac{1}{2}u\sqrt{4-u^2}+C\\&=6\sin^{-1}\left(\frac{x-2}{2}\right)-4\sqrt{4x-x^2}-\left(\frac{x-2}{2}\right)\sqrt{4x-x^2}+C\end{aligned}$$



27. $x^2 + 2x + 2 = (x+1)^2 + 1$. Let $u = x + 1$, $du = dx$. Then

$$\int \frac{dx}{(x^2+2x+2)^2} = \int \frac{du}{(u^2+1)^2} = \int \frac{\sec^2\theta\, d\theta}{\sec^4\theta} \quad \left[\begin{array}{c}\text{where } u = \tan\theta,\ du = \sec^2\theta\, d\theta, \\ \text{and } u^2+1 = \sec^2\theta\end{array}\right]$$

$$= \int \cos^2\theta\, d\theta = \tfrac{1}{2}\int (1+\cos 2\theta)\, d\theta = \tfrac{1}{2}(\theta + \sin\theta\cos\theta) + C$$

$$= \frac{1}{2}\left[\tan^{-1} u + \frac{u}{1+u^2}\right] + C = \frac{1}{2}\left[\tan^{-1}(x+1) + \frac{x+1}{x^2+2x+2}\right] + C$$

28. $5 - 4x - x^2 = -(x^2+4x+4) + 9 = 9 - (x+2)^2$. Let $u = x + 2 \;\Rightarrow\; du = dx$. Then

$$\int \frac{dx}{(5-4x-x^2)^{5/2}} = \int \frac{du}{(9-u^2)^{5/2}} = \int \frac{3\cos\theta\, d\theta}{(3\cos\theta)^5} \quad \left[\begin{array}{c}\text{where } u = 3\sin\theta,\ du = 3\cos\theta\, d\theta, \\ \text{and } \sqrt{9-u^2} = 3\cos\theta\end{array}\right]$$

$$= \tfrac{1}{81}\int \sec^4\theta\, d\theta = \tfrac{1}{81}\int (\tan^2\theta + 1)\sec^2\theta\, d\theta = \tfrac{1}{81}\left[\tfrac{1}{3}\tan^3\theta + \tan\theta\right] + C$$

$$= \frac{1}{243}\left[\frac{u^3}{(9-u^2)^{3/2}} + \frac{3u}{\sqrt{9-u^2}}\right] + C = \frac{1}{243}\left[\frac{(x+2)^3}{(5-4x-x^2)^{3/2}} + \frac{3(x+2)}{\sqrt{5-4x-x^2}}\right] + C$$

29. Let $u = x^2$, $du = 2x\, dx$. Then

$$\int x\sqrt{1-x^4}\, dx = \int \sqrt{1-u^2}\left(\tfrac{1}{2}\, du\right) = \tfrac{1}{2}\int \cos\theta \cdot \cos\theta\, d\theta \quad \left[\begin{array}{c}\text{where } u = \sin\theta,\ du = \cos\theta\, d\theta, \\ \text{and } \sqrt{1-u^2} = \cos\theta\end{array}\right]$$

$$= \tfrac{1}{2}\int \tfrac{1}{2}(1+\cos 2\theta)d\theta = \tfrac{1}{4}\theta + \tfrac{1}{8}\sin 2\theta + C = \tfrac{1}{4}\theta + \tfrac{1}{4}\sin\theta\cos\theta + C$$

$$= \tfrac{1}{4}\sin^{-1} u + \tfrac{1}{4}u\sqrt{1-u^2} + C = \tfrac{1}{4}\sin^{-1}(x^2) + \tfrac{1}{4}x^2\sqrt{1-x^4} + C$$

30. Let $u = \sin t$, $du = \cos t\, dt$. Then

$$\int_0^{\pi/2} \frac{\cos t}{\sqrt{1+\sin^2 t}}\, dt = \int_0^1 \frac{1}{\sqrt{1+u^2}}\, du = \int_0^{\pi/4} \frac{1}{\sec\theta}\sec^2\theta\, d\theta \quad \left[\begin{array}{c}\text{where } u = \tan\theta,\ du = \sec^2\theta\, d\theta, \\ \text{and } \sqrt{1+u^2} = \sec\theta\end{array}\right]$$

$$= \int_0^{\pi/4} \sec\theta\, d\theta = \Big[\ln|\sec\theta + \tan\theta|\Big]_0^{\pi/4} \quad \text{[by (1) in Section 8.2]}$$

$$= \ln(\sqrt{2}+1) - \ln(1+0) = \ln(\sqrt{2}+1)$$

31. (a) Let $x = a\tan\theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Then $\sqrt{x^2+a^2} = a\sec\theta$ and

$$\int \frac{dx}{\sqrt{x^2+a^2}} = \int \frac{a\sec^2\theta\, d\theta}{a\sec\theta} = \int \sec\theta\, d\theta = \ln|\sec\theta + \tan\theta| + C_1 = \ln\left|\frac{\sqrt{x^2+a^2}}{a} + \frac{x}{a}\right| + C_1$$

$$= \ln\left(x + \sqrt{x^2+a^2}\right) + C \quad \text{where } C = C_1 - \ln|a|$$

(b) Let $x = a\sinh t$, so that $dx = a\cosh t\, dt$ and $\sqrt{x^2+a^2} = a\cosh t$. Then

$$\int \frac{dx}{\sqrt{x^2+a^2}} = \int \frac{a\cosh t\, dt}{a\cosh t} = t + C = \sinh^{-1}\frac{x}{a} + C.$$

32. (a) Let $x = a\tan\theta$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Then

$$I = \int \frac{x^2}{(x^2+a^2)^{3/2}}\, dx = \int \frac{a^2\tan^2\theta}{a^3\sec^3\theta} a\sec^2\theta\, d\theta = \int \frac{\tan^2\theta}{\sec\theta}\, d\theta = \int \frac{\sec^2\theta - 1}{\sec\theta}\, d\theta$$

$$= \int (\sec\theta - \cos\theta)\, d\theta = \ln|\sec\theta + \tan\theta| - \sin\theta + C$$

$$= \ln\left|\frac{\sqrt{x^2+a^2}}{a} + \frac{x}{a}\right| - \frac{x}{\sqrt{x^2+a^2}} + C = \ln\left(x + \sqrt{x^2+a^2}\right) - \frac{x}{\sqrt{x^2+a^2}} + C_1$$