8 ☐ TECHNIQUES OF INTEGRATION

8.1 Integration by Parts

1. Let $u = \ln x$, $dv = x\,dx \Rightarrow du = dx/x$, $v = \frac{1}{2}x^2$. Then by Equation 2, $\int u\,dv = uv - \int v\,du$,

$$\int x \ln x\,dx = \tfrac{1}{2}x^2 \ln x - \int \tfrac{1}{2}x^2 (dx/x) = \tfrac{1}{2}x^2 \ln x - \tfrac{1}{2}\int x\,dx = \tfrac{1}{2}x^2 \ln x - \tfrac{1}{2}\cdot\tfrac{1}{2}x^2 + C$$
$$= \tfrac{1}{2}x^2 \ln x - \tfrac{1}{4}x^2 + C$$

2. Let $u = \theta$, $dv = \sec^2\theta\,d\theta \Rightarrow du = d\theta$, $v = \tan\theta$. Then
$\int \theta \sec^2\theta\,d\theta = \theta\tan\theta - \int \tan\theta\,d\theta = \theta\tan\theta - \ln|\sec\theta| + C$.

3. Let $u = x$, $dv = \cos 5x\,dx \Rightarrow du = dx$, $v = \frac{1}{5}\sin 5x$. Then by Equation 2,
$\int x\cos 5x\,dx = \frac{1}{5}x\sin 5x - \int \frac{1}{5}\sin 5x\,dx = \frac{1}{5}x\sin 5x + \frac{1}{25}\cos 5x + C$.

4. Let $u = x$, $dv = e^{-x}\,dx \Rightarrow du = dx$, $v = -e^{-x}$. Then
$\int xe^{-x}\,dx = -xe^{-x} + \int e^{-x}\,dx = -xe^{-x} - e^{-x} + C$.

5. Let $u = r$, $dv = e^{r/2}\,dr \Rightarrow du = dr$, $v = 2e^{r/2}$. Then
$\int re^{r/2}\,dr = 2re^{r/2} - \int 2e^{r/2}\,dr = 2re^{r/2} - 4e^{r/2} + C$.

6. Let $u = t$, $dv = \sin 2t\,dt \Rightarrow du = dt$, $v = -\frac{1}{2}\cos 2t$. Then
$\int t\sin 2t\,dt = -\frac{1}{2}t\cos 2t + \frac{1}{2}\int \cos 2t\,dt = -\frac{1}{2}t\cos 2t + \frac{1}{4}\sin 2t + C$.

7. Let $u = x^2$, $dv = \sin\pi x\,dx \Rightarrow du = 2x\,dx$ and $v = -\frac{1}{\pi}\cos\pi x$. Then
$I = \int x^2 \sin\pi x\,dx = -\frac{1}{\pi}x^2\cos\pi x + \frac{2}{\pi}\int x\cos\pi x\,dx$ $(*)$.
Next let $U = x$, $dV = \cos\pi x\,dx \Rightarrow dU = dx$, $V = \frac{1}{\pi}\sin\pi x$, so
$\int x\cos\pi x\,dx = \frac{1}{\pi}x\sin\pi x - \frac{1}{\pi}\int \sin\pi x\,dx = \frac{1}{\pi}x\sin\pi x + \frac{1}{\pi^2}\cos\pi x + C_1$. Substituting for $\int x\cos\pi x\,dx$ in $(*)$,
we get $I = -\frac{1}{\pi}x^2\cos\pi x + \frac{2}{\pi}\left(\frac{1}{\pi}x\sin\pi x + \frac{1}{\pi^2}\cos\pi x + C_1\right) = -\frac{1}{\pi}x^2\cos\pi x + \frac{2}{\pi^2}x\sin\pi x + \frac{2}{\pi^3}\cos\pi x + C$,
where $C = \frac{2}{\pi}C_1$.

8. Let $u = x^2$, $dv = \cos mx\,dx \Rightarrow du = 2x\,dx$, $v = \frac{1}{m}\sin mx$.
Then $I = \displaystyle\int x^2\cos mx\,dx = \frac{1}{m}x^2\sin mx - \frac{2}{m}\int x\sin mx\,dx$ $(*)$. Next let
$U = x$, $dV = \sin mx\,dx \Rightarrow dU = dx$, $V = -\frac{1}{m}\cos mx$, so
$\int x\sin mx\,dx = -\frac{1}{m}x\cos mx + \frac{1}{m}\int \cos mx\,dx = -\frac{1}{m}x\cos mx + \frac{1}{m^2}\sin mx + C_1$.
Substituting for $\int x\sin mx\,dx$ in $(*)$, we get
$I = \frac{1}{m}x^2\sin mx - \frac{2}{m}\left(-\frac{1}{m}x\cos mx + \frac{1}{m^2}\sin mx + C_1\right) = \frac{1}{m}x^2\sin mx + \frac{2}{m^2}x\cos mx - \frac{2}{m^3}\sin mx + C$,
where $C = -\frac{2}{m}C_1$.

9. Let $u = \ln(2x+1)$, $dv = dx \Rightarrow du = \dfrac{2}{2x+1}\,dx$, $v = x$. Then

$$\int \ln(2x+1)\,dx = x\ln(2x+1) - \int \frac{2x}{2x+1}\,dx = x\ln(2x+1) - \int \frac{(2x+1)-1}{2x+1}\,dx$$
$$= x\ln(2x+1) - \int\left(1 - \frac{1}{2x+1}\right)dx = x\ln(2x+1) - x + \tfrac{1}{2}\ln(2x+1) + C$$
$$= \tfrac{1}{2}(2x+1)\ln(2x+1) - x + C$$

10. Let $u = \sin^{-1} x$, $dv = dx \Rightarrow du = \dfrac{dx}{\sqrt{1-x^2}}$, $v = x$. Then

$\int \sin^{-1} x\,dx = x\sin^{-1} x - \displaystyle\int \frac{x}{\sqrt{1-x^2}}\,dx$. Setting $t = 1 - x^2$, we get $dt = -2x\,dx$, so

$-\displaystyle\int \frac{x\,dx}{\sqrt{1-x^2}} = -\int t^{-1/2}\left(-\tfrac{1}{2}\,dt\right) = \tfrac{1}{2}\left(2t^{1/2}\right) + C = t^{1/2} + C = \sqrt{1-x^2} + C$. Hence,

$\int \sin^{-1} x\,dx = x\sin^{-1} x + \sqrt{1-x^2} + C$.

11. Let $u = \arctan 4t$, $dv = dt \Rightarrow du = \dfrac{4}{1+(4t)^2}\,dt = \dfrac{4}{1+16t^2}\,dt$, $v = t$. Then

$$\int \arctan 4t\,dt = t\arctan 4t - \int \frac{4t}{1+16t^2}\,dt = t\arctan 4t - \frac{1}{8}\int \frac{32t}{1+16t^2}\,dt$$
$$= t\arctan 4t - \tfrac{1}{8}\ln(1+16t^2) + C$$

12. Let $u = \ln p$, $dv = p^5\,dp \Rightarrow du = \dfrac{1}{p}\,dp$, $v = \tfrac{1}{6}p^6$. Then

$\int p^5 \ln p\,dp = \tfrac{1}{6}p^6 \ln p - \tfrac{1}{6}\int p^5\,dp = \tfrac{1}{6}p^6 \ln p - \tfrac{1}{36}p^6 + C$.

13. First let $u = (\ln x)^2$, $dv = dx \Rightarrow du = 2\ln x \cdot \frac{1}{x}\,dx$, $v = x$. Then by Equation 2,
$I = \int (\ln x)^2\,dx = x(\ln x)^2 - 2\int x\ln x \cdot \frac{1}{x}\,dx = x(\ln x)^2 - 2\int \ln x\,dx$. Next let $U = \ln x$, $dV = dx \Rightarrow$
$dU = 1/x\,dx$, $V = x$ to get $\int \ln x\,dx = x\ln x - \int x\cdot(1/x)\,dx = x\ln x - \int dx = x\ln x - x + C_1$. Thus,
$I = x(\ln x)^2 - 2(x\ln x - x + C_1) = x(\ln x)^2 - 2x\ln x + 2x + C$, where $C = -2C_1$.

14. Let $u = t^3$, $dv = e^t\,dt \Rightarrow du = 3t^2\,dt$, $v = e^t$. Then $I = \int t^3 e^t\,dt = t^3 e^t - \int 3t^2 e^t\,dt$. Integrate by parts twice more with $dv = e^t\,dt$.

$$I = t^3e^t - \left(3t^2e^t - \textstyle\int 6te^t\,dt\right) = t^3e^t - 3t^2e^t + 6te^t - \textstyle\int 6e^t\,dt$$
$$= t^3e^t - 3t^2e^t + 6te^t - 6e^t + C = (t^3 - 3t^2 + 6t - 6)e^t + C$$

More generally, if $p(t)$ is a polynomial of degree n in t, then repeated integration by parts shows that

$\int p(t)\,e^t\,dt = \left[p(t) - p'(t) + p''(t) - p'''(t) + \cdots + (-1)^n\,p^{(n)}(t)\right]e^t + C$.

15. First let $u = \sin 3\theta$, $dv = e^{2\theta}\,d\theta \;\Rightarrow\; du = 3\cos 3\theta\,d\theta$, $v = \frac{1}{2}e^{2\theta}$. Then $I = \int e^{2\theta}\sin 3\theta\,d\theta = \frac{1}{2}e^{2\theta}\sin 3\theta - \frac{3}{2}\int e^{2\theta}\cos 3\theta\,d\theta$. Next let $U = \cos 3\theta$, $dV = e^{2\theta}\,d\theta \;\Rightarrow\; dU = -3\sin 3\theta\,d\theta$, $V = \frac{1}{2}e^{2\theta}$ to get

$\int e^{2\theta}\cos 3\theta\,d\theta = \frac{1}{2}e^{2\theta}\cos 3\theta + \frac{3}{2}\int e^{2\theta}\sin 3\theta\,d\theta$. Substituting in the previous formula gives

$I = \frac{1}{2}e^{2\theta}\sin 3\theta - \frac{3}{4}e^{2\theta}\cos 3\theta - \frac{9}{4}\int e^{2\theta}\sin 3\theta\,d\theta = \frac{1}{2}e^{2\theta}\sin 3\theta - \frac{3}{4}e^{2\theta}\cos 3\theta - \frac{9}{4}I \;\Rightarrow$

$\frac{13}{4}I = \frac{1}{2}e^{2\theta}\sin 3\theta - \frac{3}{4}e^{2\theta}\cos 3\theta + C_1$. Hence, $I = \frac{1}{13}e^{2\theta}(2\sin 3\theta - 3\cos 3\theta) + C$, where $C = \frac{4}{13}C_1$.

16. First let $u = e^{-\theta}$, $dv = \cos 2\theta\,d\theta \;\Rightarrow\; du = -e^{-\theta}\,d\theta$, $v = \frac{1}{2}\sin 2\theta$. Then

$I = \int e^{-\theta}\cos 2\theta\,d\theta = \frac{1}{2}e^{-\theta}\sin 2\theta - \int \frac{1}{2}\sin 2\theta\left(-e^{-\theta}\,d\theta\right) = \frac{1}{2}e^{-\theta}\sin 2\theta + \frac{1}{2}\int e^{-\theta}\sin 2\theta\,d\theta$.

Next let $U = e^{-\theta}$, $dV = \sin 2\theta\,d\theta \;\Rightarrow\; dU = -e^{-\theta}\,d\theta$, $V = -\frac{1}{2}\cos 2\theta$, so

$\int e^{-\theta}\sin 2\theta\,d\theta = -\frac{1}{2}e^{-\theta}\cos 2\theta - \int\left(-\frac{1}{2}\right)\cos 2\theta\left(-e^{-\theta}\,d\theta\right) = -\frac{1}{2}e^{-\theta}\cos 2\theta - \frac{1}{2}\int e^{-\theta}\cos 2\theta\,d\theta$.

So $I = \frac{1}{2}e^{-\theta}\sin 2\theta + \frac{1}{2}\left[\left(-\frac{1}{2}e^{-\theta}\cos 2\theta\right) - \frac{1}{2}I\right] = \frac{1}{2}e^{-\theta}\sin 2\theta - \frac{1}{4}e^{-\theta}\cos 2\theta - \frac{1}{4}I \;\Rightarrow$

$\frac{5}{4}I = \frac{1}{2}e^{-\theta}\sin 2\theta - \frac{1}{4}e^{-\theta}\cos 2\theta + C_1 \;\Rightarrow$

$I = \frac{4}{5}\left(\frac{1}{2}e^{-\theta}\sin 2\theta - \frac{1}{4}e^{-\theta}\cos 2\theta + C_1\right) = \frac{2}{5}e^{-\theta}\sin 2\theta - \frac{1}{5}e^{-\theta}\cos 2\theta + C$.

17. Let $u = y$, $dv = \sinh y\,dy \;\Rightarrow\; du = dy$, $v = \cosh y$. Then

$\int y\sinh y\,dy = y\cosh y - \int\cosh y\,dy = y\cosh y - \sinh y + C$.

18. Let $u = y$, $dv = \cosh ay\,dy \;\Rightarrow\; du = dy$, $v = \dfrac{\sinh ay}{a}$. Then

$$\int y\cosh ay\,dy = \frac{y\sinh ay}{a} - \frac{1}{a}\int\sinh ay\,dy = \frac{y\sinh ay}{a} - \frac{\cosh ay}{a^2} + C.$$

19. Let $u = t$, $dv = \sin 3t\,dt \;\Rightarrow\; du = dt$, $v = -\frac{1}{3}\cos 3t$. Then

$\int_0^\pi t\sin 3t\,dt = \left[-\frac{1}{3}t\cos 3t\right]_0^\pi + \frac{1}{3}\int_0^\pi\cos 3t\,dt = \left(\frac{1}{3}\pi - 0\right) + \frac{1}{9}\left[\sin 3t\right]_0^\pi = \frac{\pi}{3}$.

20. First let $u = x^2 + 1$, $dv = e^{-x}\,dx \;\Rightarrow\; du = 2x\,dx$, $v = -e^{-x}$. By (6),

$\int_0^1 (x^2+1)e^{-x}\,dx = \left[-(x^2+1)e^{-x}\right]_0^1 + \int_0^1 2xe^{-x}\,dx = -2e^{-1} + 1 + 2\int_0^1 xe^{-x}\,dx$. Next let

$U = x$, $dV = e^{-x}\,dx \;\Rightarrow\; dU = dx$, $V = -e^{-x}$. By (6) again,

$\int_0^1 xe^{-x}\,dx = \left[-xe^{-x}\right]_0^1 + \int_0^1 e^{-x}\,dx = -e^{-1} + \left[-e^{-x}\right]_0^1 = -e^{-1} - e^{-1} + 1 = -2e^{-1} + 1$. So

$\int_0^1 (x^2+1)e^{-x}\,dx = -2e^{-1} + 1 + 2(-2e^{-1}+1) = -2e^{-1} + 1 - 4e^{-1} + 2 = -6e^{-1} + 3$.

21. Let $u = \ln x$, $dv = x^{-2}\,dx \;\Rightarrow\; du = \dfrac{1}{x}\,dx$, $v = -x^{-1}$. By (6),

$$\int_1^2 \frac{\ln x}{x^2}\,dx = \left[-\frac{\ln x}{x}\right]_1^2 + \int_1^2 x^{-2}\,dx = -\tfrac{1}{2}\ln 2 + \ln 1 + \left[-\frac{1}{x}\right]_1^2 = -\tfrac{1}{2}\ln 2 + 0 - \tfrac{1}{2} + 1 = \tfrac{1}{2} - \tfrac{1}{2}\ln 2.$$

22. Let $u = \ln t$, $dv = \sqrt{t}\,dt \;\Rightarrow\; du = dt/t$, $v = \frac{2}{3}t^{3/2}$. By Formula 6,

$\int_1^4 \sqrt{t}\ln t\,dt = \left[\frac{2}{3}t^{3/2}\ln t\right]_1^4 - \frac{2}{3}\int_1^4\sqrt{t}\,dt = \frac{2}{3}\cdot 8\cdot\ln 4 - 0 - \left[\frac{2}{3}\cdot\frac{2}{3}t^{3/2}\right]_1^4 = \frac{16}{3}\ln 4 - \frac{4}{9}(8-1) = \frac{16}{3}\ln 4 - \frac{28}{9}$.

23. Let $u = y$, $dv = \dfrac{dy}{e^{2y}} = e^{-2y}\,dy \;\Rightarrow\; du = dy$, $v = -\dfrac{1}{2}e^{-2y}$. Then

$$\int_0^1 \frac{y}{e^{2y}}\,dy = \left[-\tfrac{1}{2}ye^{-2y}\right]_0^1 + \tfrac{1}{2}\int_0^1 e^{-2y}\,dy = \left(-\tfrac{1}{2}e^{-2}+0\right) - \tfrac{1}{4}\left[e^{-2y}\right]_0^1 = -\tfrac{1}{2}e^{-2} - \tfrac{1}{4}e^{-2} + \tfrac{1}{4} = \tfrac{1}{4} - \tfrac{3}{4}e^{-2}.$$

24. Let $u = x$, $dv = \csc^2 x\,dx \;\Rightarrow\; du = dx$, $v = -\cot x$. Then

$$\int_{\pi/4}^{\pi/2} x\csc^2 x\,dx = \left[-x\cot x\right]_{\pi/4}^{\pi/2} + \int_{\pi/4}^{\pi/2}\cot x\,dx = -\tfrac{\pi}{2}\cdot 0 + \tfrac{\pi}{4}\cdot 1 + \Big[\ln|\sin x|\Big]_{\pi/4}^{\pi/2} \quad \text{[see Exercise 5.5.75]}$$

$$= \tfrac{\pi}{4} + \ln 1 - \ln\tfrac{1}{\sqrt{2}} = \tfrac{\pi}{4} + 0 - \ln 2^{-1/2} = \tfrac{\pi}{4} + \tfrac{1}{2}\ln 2$$

25. Let $u = \cos^{-1}x$, $dv = dx \;\Rightarrow\; du = -\dfrac{dx}{\sqrt{1-x^2}}$, $v = x$. Then

$$I = \int_0^{1/2}\cos^{-1}x\,dx = \left[x\cos^{-1}x\right]_0^{1/2} + \int_0^{1/2}\frac{x\,dx}{\sqrt{1-x^2}} = \tfrac{1}{2}\cdot\tfrac{\pi}{3} + \int_1^{3/4} t^{-1/2}\left[-\frac{1}{2}\,dt\right], \text{ where } t = 1-x^2$$

$\Rightarrow dt = -2x\,dx$. Thus, $I = \frac{\pi}{6} + \frac{1}{2}\int_{3/4}^{1} t^{-1/2}\,dt = \frac{\pi}{6} + \left[\sqrt{t}\right]_{3/4}^{1} = \frac{\pi}{6} + 1 - \frac{\sqrt{3}}{2} = \frac{1}{6}\left(\pi + 6 - 3\sqrt{3}\right)$.

26. Let $u = x$, $dv = 5^x\,dx \;\Rightarrow\; du = dx$, $v = (5^x/\ln 5)$. Then

$$\int_0^1 x5^x\,dx = \left[\frac{x5^x}{\ln 5}\right]_0^1 - \int_0^1 \frac{5^x}{\ln 5}\,dx = \frac{5}{\ln 5} - 0 - \frac{1}{\ln 5}\left[\frac{5^x}{\ln 5}\right]_0^1 = \frac{5}{\ln 5} - \frac{5}{(\ln 5)^2} + \frac{1}{(\ln 5)^2}$$

$$= \frac{5}{\ln 5} - \frac{4}{(\ln 5)^2}$$

27. Let $u = \ln(\sin x)$, $dv = \cos x\,dx \;\Rightarrow\; du = \dfrac{\cos x}{\sin x}\,dx$, $v = \sin x$. Then

$I = \int \cos x\ln(\sin x)\,dx = \sin x\ln(\sin x) - \int\cos x\,dx = \sin x\ln(\sin x) - \sin x + C.$

Another method: Substitute $t = \sin x$, so $dt = \cos x\,dx$. Then $I = \int \ln t\,dt = t\ln t - t + C$ (see Example 2) and so $I = \sin x\,(\ln\sin x - 1) + C$.

28. Let $u = \arctan(1/x)$, $dv = dx \;\Rightarrow\; du = \dfrac{1}{1+(1/x)^2}\cdot\dfrac{-1}{x^2}\,dx = \dfrac{-dx}{x^2+1}$, $v = x$. Then

$$\int_1^{\sqrt{3}}\arctan(1/x)\,dx = \left[x\arctan\left(\frac{1}{x}\right)\right]_1^{\sqrt{3}} + \int_1^{\sqrt{3}}\frac{x\,dx}{x^2+1} = \sqrt{3}\,\frac{\pi}{6} - 1\cdot\frac{\pi}{4} + \frac{1}{2}\Big[\ln(x^2+1)\Big]_1^{\sqrt{3}}$$

$$= \frac{\pi\sqrt{3}}{6} - \frac{\pi}{4} + \frac{1}{2}(\ln 4 - \ln 2) = \frac{\pi\sqrt{3}}{6} - \frac{\pi}{2} + \frac{1}{2}\ln\frac{4}{2} = \frac{\pi\sqrt{3}}{6} - \frac{\pi}{2} + \frac{1}{2}\ln 2$$

29. Let $w = \ln x \;\Rightarrow\; dw = dx/x$. Then $x = e^w$ and $dx = e^w\,dw$, so

$$\int\cos(\ln x)\,dx = \int e^w\cos w\,dw = \tfrac{1}{2}e^w(\sin w + \cos w) + C \quad \text{[by the method of Example 4]}$$

$$= \tfrac{1}{2}x\left[\sin(\ln x) + \cos(\ln x)\right] + C$$

30. Let $u = r^2$, $dv = \dfrac{r}{\sqrt{4+r^2}}\,dr \;\Rightarrow\; du = 2r\,dr$, $v = \sqrt{4+r^2}$. By (6),

$$\int_0^1 \frac{r^3}{\sqrt{4+r^2}}\,dr = \left[r^2\sqrt{4+r^2}\right]_0^1 - 2\int_0^1 r\sqrt{4+r^2}\,dr = \sqrt{5} - \tfrac{2}{3}\left[\left(4+r^2\right)^{3/2}\right]_0^1$$

$$= \sqrt{5} - \tfrac{2}{3}(5)^{3/2} + \tfrac{2}{3}(8) = \sqrt{5}\left(1 - \tfrac{10}{3}\right) + \tfrac{16}{3} = \tfrac{16}{3} - \tfrac{7}{3}\sqrt{5}$$

31. Let $u = (\ln x)^2$, $dv = x^4\,dx \;\Rightarrow\; du = 2\dfrac{\ln x}{x}\,dx$, $v = \dfrac{x^5}{5}$. By (6),

$$\int_1^2 x^4(\ln x)^2\,dx = \left[\frac{x^5}{5}(\ln x)^2\right]_1^2 - 2\int_1^2 \frac{x^4}{5}\ln x\,dx = \tfrac{32}{5}(\ln 2)^2 - 0 - 2\int_1^2 \frac{x^4}{5}\ln x\,dx.$$

Let $U = \ln x$, $dV = \dfrac{x^4}{5}\,dx \;\Rightarrow\; dU = \dfrac{1}{x}\,dx$, $V = \dfrac{x^5}{25}$.

Then $\displaystyle\int_1^2 \frac{x^4}{5}\ln x\,dx = \left[\frac{x^5}{25}\ln x\right]_1^2 - \int_1^2 \frac{x^4}{25}\,dx = \tfrac{32}{25}\ln 2 - 0 - \left[\frac{x^5}{125}\right]_1^2 = \tfrac{32}{25}\ln 2 - \left(\tfrac{32}{125} - \tfrac{1}{125}\right).$

So $\int_1^2 x^4(\ln x)^2\,dx = \frac{32}{5}(\ln 2)^2 - 2\left(\frac{32}{25}\ln 2 - \frac{31}{125}\right) = \frac{32}{5}(\ln 2)^2 - \frac{64}{25}\ln 2 + \frac{62}{125}$.

32. Let $u = \sin(t-s)$, $dv = e^s\,ds \;\Rightarrow\; du = -\cos(t-s)\,ds$, $v = e^s$. Then
$I = \int_0^t e^s \sin(t-s)\,ds = \left[e^s \sin(t-s)\right]_0^t + \int_0^t e^s \cos(t-s)\,ds = e^t \sin 0 - e^0 \sin t + I_1$. For
I_1, let $U = \cos(t-s)$, $dV = e^s\,ds \;\Rightarrow\; dU = \sin(t-s)\,ds$, $V = e^s$. So
$I_1 = \left[e^s \cos(t-s)\right]_0^t - \int_0^t e^s \sin(t-s)ds = e^t \cos 0 - e^0 \cos t - I$. Thus, $I = -\sin t + e^t - \cos t - I \;\Rightarrow$
$2I = e^t - \cos t - \sin t \;\Rightarrow\; I = \frac{1}{2}\left(e^t - \cos t - \sin t\right)$.

33. Let $w = \sqrt{x}$, so that $x = w^2$ and $dx = 2w\,dw$. Thus, $\int \sin\sqrt{x}\,dx = \int 2w \sin w\,dw$. Now use parts with $u = 2w$, $dv = \sin w\,dw$, $du = 2\,dw$, $v = -\cos w$ to get

$$\begin{aligned}\int 2w\sin w\,dw &= -2w\cos w + \int 2\cos w\,dw = -2w\cos w + 2\sin w + C \\ &= -2\sqrt{x}\cos\sqrt{x} + 2\sin\sqrt{x} + C = 2\left(\sin\sqrt{x} - \sqrt{x}\cos\sqrt{x}\right) + C\end{aligned}$$

34. Let $w = \sqrt{x}$, so that $x = w^2$ and $dx = 2w\,dw$. Thus, $\int_1^4 e^{\sqrt{x}}\,dx = \int_1^2 e^w 2w\,dw$. Now use parts with $u = 2w$, $dv = e^w\,dw$, $du = 2\,dw$, $v = e^w$ to get $\int_1^2 e^w 2w\,dw = \left[2we^w\right]_1^2 - 2\int_1^2 e^w\,dw = 4e^2 - 2e - 2(e^2 - e) = 2e^2$.

35. Let $x = \theta^2$, so that $dx = 2\theta\,d\theta$. Thus, $\int_{\sqrt{\pi/2}}^{\sqrt{\pi}} \theta^3\cos(\theta^2)\,d\theta = \int_{\sqrt{\pi/2}}^{\sqrt{\pi}} \theta^2\cos(\theta^2)\cdot\frac{1}{2}(2\theta\,d\theta) = \frac{1}{2}\int_{\pi/2}^{\pi} x\cos x\,dx$.
Now use parts with $u = x$, $dv = \cos x\,dx$, $du = dx$, $v = \sin x$ to get

$$\begin{aligned}\tfrac{1}{2}\int_{\pi/2}^{\pi} x\cos x\,dx &= \tfrac{1}{2}\left(\left[x\sin x\right]_{\pi/2}^{\pi} - \int_{\pi/2}^{\pi}\sin x\,dx\right) = \tfrac{1}{2}\left[x\sin x + \cos x\right]_{\pi/2}^{\pi} \\ &= \tfrac{1}{2}(\pi\sin\pi + \cos\pi) - \tfrac{1}{2}\left(\tfrac{\pi}{2}\sin\tfrac{\pi}{2} + \cos\tfrac{\pi}{2}\right) = \tfrac{1}{2}(\pi\cdot 0 - 1) - \tfrac{1}{2}\left(\tfrac{\pi}{2}\cdot 1 + 0\right) = -\tfrac{1}{2} - \tfrac{\pi}{4}\end{aligned}$$

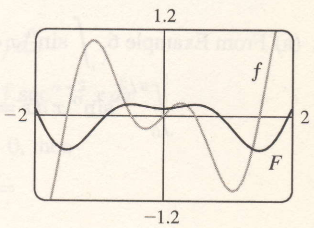
36. $\displaystyle\int x^5 e^{x^2}\,dx = \int (x^2)^2 e^{x^2} x\,dx = \int t^2 e^t \tfrac{1}{2}\,dt \quad$ [where $t = x^2 \;\Rightarrow\; \frac{1}{2}\,dt = x\,dx$]
$\quad = \frac{1}{2}\left(t^2 - 2t + 2\right)e^t + C \quad$ [by Example 3] $\; = \frac{1}{2}\left(x^4 - 2x^2 + 2\right)e^{x^2} + C$

In Exercises 37–40, let $f(x)$ denote the integrand and $F(x)$ its antiderivative (with $C = 0$).

37. Let $u = x$, $dv = \cos\pi x\,dx \;\Rightarrow\; du = dx$, $v = (\sin\pi x)/\pi$. Then

$$\int x\cos\pi x\,dx = x\cdot\frac{\sin\pi x}{\pi} - \int\frac{\sin\pi x}{\pi}\,dx = \frac{x\sin\pi x}{\pi} + \frac{\cos\pi x}{\pi^2} + C.$$

We see from the graph that this is reasonable, since F has extreme values where f is 0.



(b) Using $n = 3$ in part (a), we have $\int_0^{\pi/2} \sin^3 x\,dx = \frac{2}{3}\int_0^{\pi/2} \sin x\,dx = \left[-\frac{2}{3}\cos x\right]_0^{\pi/2} = \frac{2}{3}$.

Using $n = 5$ in part (a), we have $\int_0^{\pi/2} \sin^5 x\,dx = \frac{4}{5}\int_0^{\pi/2} \sin^3 x\,dx = \frac{4}{5}\cdot\frac{2}{3} = \frac{8}{15}$.

(c) The formula holds for $n = 1$ (that is, $2n + 1 = 3$) by (b). Assume it holds for some $k \geq 1$. Then

$\displaystyle\int_0^{\pi/2} \sin^{2k+1} x\,dx = \frac{2\cdot 4\cdot 6\cdot\cdots\cdot(2k)}{3\cdot 5\cdot 7\cdot\cdots\cdot(2k+1)}$. By Example 6,

$$\begin{aligned}\int_0^{\pi/2} \sin^{2k+3} x\,dx &= \frac{2k+2}{2k+3}\int_0^{\pi/2} \sin^{2k+1} x\,dx = \frac{2k+2}{2k+3}\cdot\frac{2\cdot 4\cdot 6\cdot\cdots\cdot(2k)}{3\cdot 5\cdot 7\cdot\cdots\cdot(2k+1)} \\ &= \frac{2\cdot 4\cdot 6\cdot\cdots\cdot(2k)[2(k+1)]}{3\cdot 5\cdot 7\cdot\cdots\cdot(2k+1)[2(k+1)+1]},\end{aligned}$$

so the formula holds for $n = k + 1$. By induction, the formula holds for all $n \geq 1$.

44. Using Exercise 43(a), we see that the formula holds for $n = 1$, because $\int_0^{\pi/2} \sin^2 x\,dx = \frac{1}{2}\int_0^{\pi/2} 1\,dx = \frac{1}{2}\left[x\right]_0^{\pi/2} = \frac{1}{2}\cdot\frac{\pi}{2}$.

Now assume it holds for some $k \geq 1$. Then $\displaystyle\int_0^{\pi/2} \sin^{2k} x\,dx = \frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2k-1)}{2\cdot 4\cdot 6\cdot\cdots\cdot(2k)}\frac{\pi}{2}$. By Exercise 43(a),

$$\begin{aligned}\int_0^{\pi/2} \sin^{2(k+1)} x\,dx &= \frac{2k+1}{2k+2}\int_0^{\pi/2} \sin^{2k} x\,dx = \frac{2k+1}{2k+2}\cdot\frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2k-1)}{2\cdot 4\cdot 6\cdot\cdots\cdot(2k)}\frac{\pi}{2} \\ &= \frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2k-1)(2k+1)}{2\cdot 4\cdot 6\cdot\cdots\cdot(2k)(2k+2)}\cdot\frac{\pi}{2},\end{aligned}$$

so the formula holds for $n = k + 1$. By induction, the formula holds for all $n \geq 1$.

45. Let $u = (\ln x)^n$, $dv = dx$ $\Rightarrow$ $du = n(\ln x)^{n-1}(dx/x)$, $v = x$. By Equation 2, $\int(\ln x)^n\,dx = x(\ln x)^n - \int nx(\ln x)^{n-1}(dx/x) = x(\ln x)^n - n\int(\ln x)^{n-1}\,dx$.

46. Let $u = x^n$, $dv = e^x\,dx$ $\Rightarrow$ $du = nx^{n-1}\,dx$, $v = e^x$. By Equation 2, $\int x^n e^x\,dx = x^n e^x - n\int x^{n-1}e^x\,dx$.

47. Let $u = (x^2 + a^2)^n$, $dv = dx$ $\Rightarrow$ $du = n(x^2 + a^2)^{n-1}\,2x\,dx$, $v = x$. Then

$$\begin{aligned}\int(x^2+a^2)^n\,dx &= x(x^2+a^2)^n - 2n\int x^2(x^2+a^2)^{n-1}\,dx \\ &= x(x^2+a^2)^n - 2n\left[\int(x^2+a^2)^n\,dx - a^2\int(x^2+a^2)^{n-1}\,dx\right] \quad [\text{since } x^2 = (x^2+a^2) - a^2]\end{aligned}$$

$\Rightarrow$ $(2n+1)\int(x^2+a^2)^n\,dx = x(x^2+a^2)^n + 2na^2\int(x^2+a^2)^{n-1}\,dx$, and

$$\int(x^2+a^2)^n\,dx = \frac{x(x^2+a^2)^n}{2n+1} + \frac{2na^2}{2n+1}\int(x^2+a^2)^{n-1}\,dx \quad [\text{provided } 2n+1 \neq 0].$$

48. Let $u = \sec^{n-2} x$, $dv = \sec^2 x\,dx$ $\Rightarrow$ $du = (n-2)\sec^{n-3} x\sec x\tan x\,dx$, $v = \tan x$. Then by Equation 2,

$$\begin{aligned}\int\sec^n x\,dx &= \tan x\sec^{n-2} x - (n-2)\int\sec^{n-2} x\tan^2 x\,dx \\ &= \tan x\sec^{n-2} x - (n-2)\int\sec^{n-2} x(\sec^2 x - 1)\,dx \\ &= \tan x\sec^{n-2} x - (n-2)\int\sec^n x\,dx + (n-2)\int\sec^{n-2} x\,dx\end{aligned}$$

so $(n-1)\int\sec^n x\,dx = \tan x\sec^{n-2} x + (n-2)\int\sec^{n-2} x\,dx$. If $n - 1 \neq 0$, then

$$\int\sec^n x\,dx = \frac{\tan x\sec^{n-2} x}{n-1} + \frac{n-2}{n-1}\int\sec^{n-2} x\,dx.$$