5. $\int \cos^5 x \sin^4 x\,dx = \int \cos^4 x \sin^4 x \cos x\,dx = \int (1-\sin^2 x)^2 \sin^4 x \cos x\,dx \overset{s}{=} \int (1-u^2)^2 u^4\,du$

$= \int (1-2u^2+u^4)\,u^4\,du = \int (u^4-2u^6+u^8)\,du = \frac{1}{5}u^5 - \frac{2}{7}u^7 + \frac{1}{9}u^9 + C$

$= \frac{1}{5}\sin^5 x - \frac{2}{7}\sin^7 x + \frac{1}{9}\sin^9 x + C$

6. $\int \sin^3(mx)\,dx = \int (1-\cos^2 mx)\sin mx\,dx = -\frac{1}{m}\int (1-u^2)\,du \quad [u = \cos mx,\ du = -m\sin mx\,dx]$

$= -\frac{1}{m}\left(u - \frac{1}{3}u^3\right) + C = -\frac{1}{m}\left(\cos mx - \frac{1}{3}\cos^3 mx\right) + C$

$= \frac{1}{3m}\cos^3 mx - \frac{1}{m}\cos mx + C$

7. $\int_0^{\pi/2} \cos^2\theta\,d\theta = \int_0^{\pi/2} \frac{1}{2}(1+\cos 2\theta)\,d\theta$ [half-angle identity]

$= \frac{1}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\pi/2} = \frac{1}{2}\left[\left(\frac{\pi}{2}+0\right) - (0+0)\right] = \frac{\pi}{4}$

8. $\int_0^{\pi/2} \sin^2(2\theta)\,d\theta = \int_0^{\pi/2} \frac{1}{2}(1-\cos 4\theta)d\theta = \frac{1}{2}\left[\theta - \frac{1}{4}\sin 4\theta\right]_0^{\pi/2} = \frac{1}{2}\left[\left(\frac{\pi}{2}-0\right) - (0-0)\right] = \frac{\pi}{4}$

9. $\int_0^{\pi} \sin^4(3t)\,dt = \int_0^{\pi} \left[\sin^2(3t)\right]^2 dt = \int_0^{\pi} \left[\frac{1}{2}(1-\cos 6t)\right]^2 dt = \frac{1}{4}\int_0^{\pi} (1 - 2\cos 6t + \cos^2 6t)\,dt$

$= \frac{1}{4}\int_0^{\pi} \left[1 - 2\cos 6t + \frac{1}{2}(1+\cos 12t)\right] dt = \frac{1}{4}\int_0^{\pi} \left(\frac{3}{2} - 2\cos 6t + \frac{1}{2}\cos 12t\right) dt$

$= \frac{1}{4}\left[\frac{3}{2}t - \frac{1}{3}\sin 6t + \frac{1}{24}\sin 12t\right]_0^{\pi} = \frac{1}{4}\left[\left(\frac{3\pi}{2} - 0 + 0\right) - (0-0+0)\right] = \frac{3\pi}{8}$

10. $\int_0^{\pi} \cos^6\theta\,d\theta = \int_0^{\pi} (\cos^2\theta)^3\,d\theta = \int_0^{\pi} \left[\frac{1}{2}(1+\cos 2\theta)\right]^3 d\theta = \frac{1}{8}\int_0^{\pi} (1 + 3\cos 2\theta + 3\cos^2 2\theta + \cos^3 2\theta)\,d\theta$

$= \frac{1}{8}\left[\theta + \frac{3}{2}\sin 2\theta\right]_0^{\pi} + \frac{1}{8}\int_0^{\pi} \left[\frac{3}{2}(1+\cos 4\theta)\right] d\theta + \frac{1}{8}\int_0^{\pi} \left[(1-\sin^2 2\theta)\cos 2\theta\right] d\theta$

$= \frac{1}{8}\pi + \frac{3}{16}\left[\theta + \frac{1}{4}\sin 4\theta\right]_0^{\pi} + \frac{1}{8}\int_0^{0} (1-u^2)\left(\frac{1}{2}du\right) \quad [u = \sin 2\theta,\ du = 2\cos 2\theta\,d\theta]$

$= \frac{\pi}{8} + \frac{3\pi}{16} + 0 = \frac{5\pi}{16}$

11. $\int (1+\cos\theta)^2\,d\theta = \int (1 + 2\cos\theta + \cos^2\theta)\,d\theta = \theta + 2\sin\theta + \frac{1}{2}\int (1+\cos 2\theta)\,d\theta$

$= \theta + 2\sin\theta + \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta + C = \frac{3}{2}\theta + 2\sin\theta + \frac{1}{4}\sin 2\theta + C$

12. Let $u = x$, $dv = \cos^2 x\,dx \quad\Rightarrow\quad du = dx$, $v = \int \cos^2 x\,dx = \int \frac{1}{2}(1+\cos 2x)dx = \frac{1}{2}x + \frac{1}{4}\sin 2x$, so

$\int x\cos^2 x\,dx = x\left(\frac{1}{2}x + \frac{1}{4}\sin 2x\right) - \int \left(\frac{1}{2}x + \frac{1}{4}\sin 2x\right) dx = \frac{1}{2}x^2 + \frac{1}{4}x\sin 2x - \frac{1}{4}x^2 + \frac{1}{8}\cos 2x + C$

$= \frac{1}{4}x^2 + \frac{1}{4}x\sin 2x + \frac{1}{8}\cos 2x + C$

13. $\int_0^{\pi/4} \sin^4 x\cos^2 x\,dx = \int_0^{\pi/4} \sin^2 x\,(\sin x\cos x)^2\,dx = \int_0^{\pi/4} \frac{1}{2}(1-\cos 2x)\left(\frac{1}{2}\sin 2x\right)^2 dx$

$= \frac{1}{8}\int_0^{\pi/4} (1-\cos 2x)\sin^2 2x\,dx = \frac{1}{8}\int_0^{\pi/4} \sin^2 2x\,dx - \frac{1}{8}\int_0^{\pi/4} \sin^2 2x\cos 2x\,dx$

$= \frac{1}{16}\int_0^{\pi/4} (1-\cos 4x)\,dx - \frac{1}{16}\left[\frac{1}{3}\sin^3 2x\right]_0^{\pi/4} = \frac{1}{16}\left[x - \frac{1}{4}\sin 4x - \frac{1}{3}\sin^3 2x\right]_0^{\pi/4}$

$= \frac{1}{16}\left(\frac{\pi}{4} - 0 - \frac{1}{3}\right) = \frac{1}{192}(3\pi - 4)$

14. $\int_0^{\pi/2} \sin^2 x\cos^2 x\,dx = \int_0^{\pi/2} \frac{1}{4}(4\sin^2 x\cos^2 x)\,dx = \int_0^{\pi/2} \frac{1}{4}(2\sin x\cos x)^2 dx = \frac{1}{4}\int_0^{\pi/2} \sin^2 2x\,dx$

$= \frac{1}{4}\int_0^{\pi/2} \frac{1}{2}(1-\cos 4x)\,dx = \frac{1}{8}\int_0^{\pi/2} (1-\cos 4x)\,dx = \frac{1}{8}\left[x - \frac{1}{4}\sin 4x\right]_0^{\pi/2}$

$= \frac{1}{8}\left(\frac{\pi}{2}\right) = \frac{\pi}{16}$

15. $\int \sin^3 x\sqrt{\cos x}\,dx = \int (1-\cos^2 x)\sqrt{\cos x}\sin x\,dx \overset{c}{=} \int (1-u^2)\,u^{1/2}(-du) = \int \left(u^{5/2} - u^{1/2}\right) du$

$= \frac{2}{7}u^{7/2} - \frac{2}{3}u^{3/2} + C = \frac{2}{7}(\cos x)^{7/2} - \frac{2}{3}(\cos x)^{3/2} + C$

$= \left(\frac{2}{7}\cos^3 x - \frac{2}{3}\cos x\right)\sqrt{\cos x} + C$

16. Let $u = \sin\theta$. Then $du = \cos\theta\, d\theta$ and

$$\int \cos\theta \cos^5(\sin\theta)\, d\theta = \int \cos^5 u\, du = \int (\cos^2 u)^2 \cos u\, du = \int (1 - \sin^2 u)^2 \cos u\, du$$
$$= \int (1 - 2\sin^2 u + \sin^4 u) \cos u\, du = I$$

Now let $x = \sin u$. Then $dx = \cos u\, du$ and

$$I = \int (1 - 2x^2 + x^4)\, dx = x - \tfrac{2}{3}x^3 + \tfrac{1}{5}x^5 + C = \sin u - \tfrac{2}{3}\sin^3 u + \tfrac{1}{5}\sin^5 u + C$$
$$= \sin(\sin\theta) - \tfrac{2}{3}\sin^3(\sin\theta) + \tfrac{1}{5}\sin^5(\sin\theta) + C$$

17. $$\int \cos^2 x \tan^3 x\, dx = \int \frac{\sin^3 x}{\cos x}\, dx \overset{c}{=} \int \frac{(1 - u^2)(-du)}{u} = \int \left[\frac{-1}{u} + u\right] du$$
$$= -\ln|u| + \tfrac{1}{2}u^2 + C = \tfrac{1}{2}\cos^2 x - \ln|\cos x| + C$$

18. $$\int \cot^5\theta \sin^4\theta\, d\theta = \int \frac{\cos^5\theta}{\sin^5\theta} \sin^4\theta\, d\theta = \int \frac{\cos^5\theta}{\sin\theta}\, d\theta = \int \frac{\cos^4\theta}{\sin\theta} \cos\theta\, d\theta = \int \frac{(1 - \sin^2\theta)^2}{\sin\theta} \cos\theta\, d\theta$$
$$\overset{s}{=} \int \frac{(1 - u^2)^2}{u}\, du = \int \frac{1 - 2u^2 + u^4}{u}\, du = \int \left(\frac{1}{u} - 2u + u^3\right) du$$
$$= \ln|u| - u^2 + \tfrac{1}{4}u^4 + C = \ln|\sin\theta| - \sin^2\theta + \tfrac{1}{4}\sin^4\theta + C$$

19. $$\int \frac{1 - \sin x}{\cos x}\, dx = \int (\sec x - \tan x)\, dx = \ln|\sec x + \tan x| - \ln|\sec x| + C \quad \left[\begin{array}{c}\text{by (1) and the boxed}\\ \text{formula above it}\end{array}\right]$$
$$= \ln|(\sec x + \tan x)\cos x| + C = \ln|1 + \sin x| + C$$
$$= \ln(1 + \sin x) + C \quad \text{since } 1 + \sin x \geq 0$$

Or: $$\int \frac{1 - \sin x}{\cos x}\, dx = \int \frac{1 - \sin x}{\cos x} \cdot \frac{1 + \sin x}{1 + \sin x}\, dx = \int \frac{(1 - \sin^2 x)\, dx}{\cos x\,(1 + \sin x)} = \int \frac{\cos x\, dx}{1 + \sin x}$$
$$= \int \frac{dw}{w} \quad [\text{where } w = 1 + \sin x,\ dw = \cos x\, dx]$$
$$= \ln|w| + C = \ln|1 + \sin x| + C = \ln(1 + \sin x) + C$$

20. $\int \cos^2 x \sin 2x\, dx = 2\int \cos^3 x \sin x\, dx \overset{c}{=} -2\int u^3\, du = -\tfrac{1}{2}u^4 + C = -\tfrac{1}{2}\cos^4 x + C$

21. Let $u = \tan x$, $du = \sec^2 x\, dx$. Then $\int \sec^2 x \tan x\, dx = \int u\, du = \tfrac{1}{2}u^2 + C = \tfrac{1}{2}\tan^2 x + C$.

Or: Let $v = \sec x$, $dv = \sec x \tan x\, dx$. Then $\int \sec^2 x \tan x\, dx = \int v\, dv = \tfrac{1}{2}v^2 + C = \tfrac{1}{2}\sec^2 x + C$.

22. $\int_0^{\pi/2} \sec^4(t/2)\, dt = \int_0^{\pi/4} \sec^4 x\,(2\, dx) \quad [x = t/2,\ dx = \tfrac{1}{2}\, dt] \quad = 2\int_0^{\pi/4} \sec^2 x(1 + \tan^2 x)\, dx$

$= 2\int_0^1 (1 + u^2)\, du \quad [u = \tan x,\ du = \sec^2 x\, dx] \quad = 2\left[u + \tfrac{1}{3}u^3\right]_0^1 = 2\left(1 + \tfrac{1}{3}\right) = \tfrac{8}{3}$

23. $\int \tan^2 x\, dx = \int (\sec^2 x - 1)\, dx = \tan x - x + C$

24. $\int \tan^4 x\, dx = \int \tan^2 x\,(\sec^2 x - 1)\, dx = \int \tan^2 x \sec^2 x\, dx - \int \tan^2 x\, dx = \tfrac{1}{3}\tan^3 x - \tan x + x + C$

(Set $u = \tan x$ in the first integral and use Exercise 23 for the second.)

25. $\int \sec^6 t\, dt = \int \sec^4 t \cdot \sec^2 t\, dt = \int (\tan^2 t + 1)^2 \sec^2 t\, dt = \int (u^2 + 1)^2\, du \quad [u = \tan t,\ du = \sec^2 t\, dt]$

$= \int (u^4 + 2u^2 + 1)\, du = \tfrac{1}{5}u^5 + \tfrac{2}{3}u^3 + u + C = \tfrac{1}{5}\tan^5 t + \tfrac{2}{3}\tan^3 t + \tan t + C$

26. $\int_0^{\pi/4} \sec^4\theta \tan^4\theta\, d\theta = \int_0^{\pi/4} (\tan^2\theta + 1)\tan^4\theta \sec^2\theta\, d\theta = \int_0^1 (u^2 + 1)u^4\, du \quad [u = \tan\theta,\ du = \sec^2\theta\, d\theta]$

$= \int_0^1 (u^6 + u^4)\, du = \left[\tfrac{1}{7}u^7 + \tfrac{1}{5}u^5\right]_0^1 = \tfrac{1}{7} + \tfrac{1}{5} = \tfrac{12}{35}$

27. $\int_0^{\pi/3} \tan^5 x \sec^4 x\, dx = \int_0^{\pi/3} \tan^5 x\,(\tan^2 x + 1) \sec^2 x\, dx$

$= \int_0^{\sqrt{3}} u^5 (u^2+1)\, du \qquad [u = \tan x,\, du = \sec^2 x\, dx]$

$= \int_0^{\sqrt{3}} (u^7 + u^5)\, du = \left[\frac{1}{8}u^8 + \frac{1}{6}u^6\right]_0^{\sqrt{3}} = \frac{81}{8} + \frac{27}{6} = \frac{81}{8} + \frac{9}{2} = \frac{81}{8} + \frac{36}{8} = \frac{117}{8}$

Alternate solution:

$\int_0^{\pi/3} \tan^5 x \sec^4 x\, dx = \int_0^{\pi/3} \tan^4 x \sec^3 x \sec x \tan x\, dx = \int_0^{\pi/3} (\sec^2 x - 1)^2 \sec^3 x \sec x \tan x\, dx$

$= \int_1^2 (u^2-1)^2 u^3\, du \qquad [u = \sec x,\, du = \sec x \tan x\, dx]$

$= \int_1^2 (u^4 - 2u^2 + 1) u^3\, du = \int_1^2 (u^7 - 2u^5 + u^3)\, du$

$= \left[\frac{1}{8}u^8 - \frac{1}{3}u^6 + \frac{1}{4}u^4\right]_1^2 = \left(32 - \frac{64}{3} + 4\right) - \left(\frac{1}{8} - \frac{1}{3} + \frac{1}{4}\right) = \frac{117}{8}$

28. $\int \tan^3(2x) \sec^5(2x)\, dx = \int \tan^2(2x) \sec^4(2x) \cdot \sec(2x) \tan(2x)\, dx$

$= \int (u^2-1)u^4\left(\frac{1}{2}\,du\right) \qquad [u = \sec(2x),\, du = 2\sec(2x)\tan(2x)\, dx]$

$= \frac{1}{2}\int (u^6 - u^4)\, du = \frac{1}{14}u^7 - \frac{1}{10}u^5 + C = \frac{1}{14}\sec^7(2x) - \frac{1}{10}\sec^5(2x) + C$

29. $\int \tan^3 x \sec x\, dx = \int \tan^2 x \sec x \tan x\, dx = \int \left(\sec^2 x - 1\right) \sec x \tan x\, dx$

$= \int (u^2 - 1)\, du \qquad [u = \sec x,\, du = \sec x \tan x\, dx]$

$= \frac{1}{3}u^3 - u + C = \frac{1}{3}\sec^3 x - \sec x + C$

30. $\int_0^{\pi/3} \tan^5 x \sec^6 x\, dx = \int_0^{\pi/3} \tan^5 x \sec^4 x \sec^2 x\, dx = \int_0^{\pi/3} \tan^5 x \left(1 + \tan^2 x\right)^2 \sec^2 x\, dx$

$= \int_0^{\sqrt{3}} u^5 \left(1+u^2\right)^2\, du \quad [u = \tan x,\, du = \sec^2 x\, dx] \quad = \int_0^{\sqrt{3}} u^5 \left(1 + 2u^2 + u^4\right) du$

$= \int_0^{\sqrt{3}} \left(u^5 + 2u^7 + u^9\right) du = \left[\frac{1}{6}u^6 + \frac{1}{4}u^8 + \frac{1}{10}u^{10}\right]_0^{\sqrt{3}} = \frac{27}{6} + \frac{81}{4} + \frac{243}{10} = \frac{981}{20}$

Alternate solution:

$\int_0^{\pi/3} \tan^5 x \sec^6 x\, dx = \int_0^{\pi/3} \tan^4 x \sec^5 x \sec x \tan x\, dx = \int_0^{\pi/3} \left(\sec^2 x - 1\right)^2 \sec^5 x \sec x \tan x\, dx$

$= \int_1^2 \left(u^2 - 1\right)^2 u^5\, du \qquad [u = \sec x,\, du = \sec x \tan x\, dx]$

$= \int_1^2 \left(u^4 - 2u^2 + 1\right) u^5\, du = \int_1^2 \left(u^9 - 2u^7 + u^5\right) du$

$= \left[\frac{1}{10}u^{10} - \frac{1}{4}u^8 + \frac{1}{6}u^6\right]_1^2 = \left(\frac{512}{5} - 64 + \frac{32}{3}\right) - \left(\frac{1}{10} - \frac{1}{4} + \frac{1}{6}\right) = \frac{981}{20}$

31. $\int \tan^5 x\, dx = \int \left(\sec^2 x - 1\right)^2 \tan x\, dx = \int \sec^4 x \tan x\, dx - 2\int \sec^2 x \tan x\, dx + \int \tan x\, dx$

$= \int \sec^3 x \sec x \tan x\, dx - 2\int \tan x \sec^2 x\, dx + \int \tan x\, dx$

$= \frac{1}{4}\sec^4 x - \tan^2 x + \ln|\sec x| + C \quad \left[\text{or } \frac{1}{4}\sec^4 x - \sec^2 x + \ln|\sec x| + C\right]$

32. $\int \tan^6 ay\, dy = \int \tan^4 ay \left(\sec^2 ay - 1\right) dy = \int \tan^4 ay \sec^2 ay\, dy - \int \tan^4 ay\, dy$

$= \frac{1}{5a}\tan^5 ay - \int \tan^2 ay \left(\sec^2 ay - 1\right) dy$

$= \frac{1}{5a}\tan^5 ay - \int \tan^2 ay \sec^2 ay\, dy + \int \left(\sec^2 ay - 1\right) dy$

$= \frac{1}{5a}\tan^5 ay - \frac{1}{3a}\tan^3 ay + \frac{1}{a}\tan ay - y + C$

33. $\displaystyle\int \frac{\tan^3\theta}{\cos^4\theta}\, d\theta = \int \tan^3\theta \sec^4\theta\, d\theta = \int \tan^3\theta \cdot (\tan^2\theta + 1) \cdot \sec^2\theta\, d\theta$

$= \int u^3(u^2+1)\, du \qquad [u = \tan\theta,\, du = \sec^2\theta\, d\theta]$

$= \int (u^5 + u^3)\, du = \frac{1}{6}u^6 + \frac{1}{4}u^4 + C = \frac{1}{6}\tan^6\theta + \frac{1}{4}\tan^4\theta + C$