3. With a CAS, we type in the definition of θ, substitute in the proper values of a and b in terms of x and $\alpha = 20° = \frac{\pi}{9}$ radians, and then use the differentiation command to find the derivative. We use a numerical root finder and find that the root of the equation $d\theta/dx = 0$ is $x \approx 8.253062$, as approximated in Problem 2.

4. From the graph in Problem 2, it seems that the average value of the function on the interval $[0, 60]$ is about 0.6. We can use a CAS to approximate $\frac{1}{60} \int_0^{60} \theta(x)\, dx \approx 0.625 \approx 36°$. (The calculation is much faster if we reduce the number of digits of accuracy required.) The minimum value is $\theta(60) \approx 0.38$ and, from Problem 2, the maximum value is about 0.85.

7.6 Hyperbolic Functions

1. (a) $\sinh 0 = \frac{1}{2}(e^0 - e^0) = 0$
   (b) $\cosh 0 = \frac{1}{2}(e^0 + e^0) = \frac{1}{2}(1 + 1) = 1$

2. (a) $\tanh 0 = \dfrac{(e^0 - e^{-0})/2}{(e^0 + e^{-0})/2} = 0$
   (b) $\tanh 1 = \dfrac{e^1 - e^{-1}}{e^1 + e^{-1}} = \dfrac{e^2 - 1}{e^2 + 1} \approx 0.76159$

3. (a) $\sinh(\ln 2) = \dfrac{e^{\ln 2} - e^{-\ln 2}}{2} = \dfrac{e^{\ln 2} - (e^{\ln 2})^{-1}}{2} = \dfrac{2 - 2^{-1}}{2} = \dfrac{2 - \frac{1}{2}}{2} = \dfrac{3}{4}$
   (b) $\sinh 2 = \frac{1}{2}(e^2 - e^{-2}) \approx 3.62686$

4. (a) $\cosh 3 = \frac{1}{2}(e^3 + e^{-3}) \approx 10.06766$
   (b) $\cosh(\ln 3) = \dfrac{e^{\ln 3} + e^{-\ln 3}}{2} = \dfrac{3 + \frac{1}{3}}{2} = \dfrac{5}{3}$

5. (a) $\text{sech } 0 = \dfrac{1}{\cosh 0} = \dfrac{1}{1} = 1$
   (b) $\cosh^{-1} 1 = 0$ because $\cosh 0 = 1$.

6. (a) $\sinh 1 = \frac{1}{2}(e^1 - e^{-1}) \approx 1.17520$
   (b) Using Equation 3, we have $\sinh^{-1} 1 = \ln(1 + \sqrt{1^2 + 1}) = \ln(1 + \sqrt{2}) \approx 0.88137$.

7. $\sinh(-x) = \frac{1}{2}\left[e^{-x} - e^{-(-x)}\right] = \frac{1}{2}(e^{-x} - e^x) = -\frac{1}{2}(e^x - e^{-x}) = -\sinh x$

8. $\cosh(-x) = \frac{1}{2}\left[e^{-x} + e^{-(-x)}\right] = \frac{1}{2}(e^{-x} + e^x) = \frac{1}{2}(e^x + e^{-x}) = \cosh x$

9. $\cosh x + \sinh x = \frac{1}{2}(e^x + e^{-x}) + \frac{1}{2}(e^x - e^{-x}) = \frac{1}{2}(2e^x) = e^x$

10. $\cosh x - \sinh x = \frac{1}{2}(e^x + e^{-x}) - \frac{1}{2}(e^x - e^{-x}) = \frac{1}{2}(2e^{-x}) = e^{-x}$

11. $\sinh x \cosh y + \cosh x \sinh y = \left[\frac{1}{2}(e^x - e^{-x})\right]\left[\frac{1}{2}(e^y + e^{-y})\right] + \left[\frac{1}{2}(e^x + e^{-x})\right]\left[\frac{1}{2}(e^y - e^{-y})\right]$
    $= \frac{1}{4}\left[(e^{x+y} + e^{x-y} - e^{-x+y} - e^{-x-y}) + (e^{x+y} - e^{x-y} + e^{-x+y} - e^{-x-y})\right]$
    $= \frac{1}{4}(2e^{x+y} - 2e^{-x-y}) = \frac{1}{2}\left[e^{x+y} - e^{-(x+y)}\right] = \sinh(x + y)$

12. $\cosh x \cosh y + \sinh x \sinh y = \left[\frac{1}{2}\left(e^x + e^{-x}\right)\right]\left[\frac{1}{2}\left(e^y + e^{-y}\right)\right] + \left[\frac{1}{2}\left(e^x - e^{-x}\right)\right]\left[\frac{1}{2}\left(e^y - e^{-y}\right)\right]$

$$= \frac{1}{4}\left[\left(e^{x+y} + e^{x-y} + e^{-x+y} + e^{-x-y}\right) + \left(e^{x+y} - e^{x-y} - e^{-x+y} + e^{-x-y}\right)\right]$$

$$= \frac{1}{4}\left(2e^{x+y} + 2e^{-x-y}\right) = \frac{1}{2}\left[e^{x+y} + e^{-(x+y)}\right] = \cosh(x+y)$$

13. Divide both sides of the identity $\cosh^2 x - \sinh^2 x = 1$ by $\sinh^2 x$:

$$\frac{\cosh^2 x}{\sinh^2 x} - \frac{\sinh^2 x}{\sinh^2 x} = \frac{1}{\sinh^2 x} \quad \Leftrightarrow \quad \coth^2 x - 1 = \operatorname{csch}^2 x.$$

14. $\tanh(x+y) = \dfrac{\sinh(x+y)}{\cosh(x+y)} = \dfrac{\sinh x \cosh y + \cosh x \sinh y}{\cosh x \cosh y + \sinh x \sinh y} = \dfrac{\dfrac{\sinh x \cosh y}{\cosh x \cosh y} + \dfrac{\cosh x \sinh y}{\cosh x \cosh y}}{\dfrac{\cosh x \cosh y}{\cosh x \cosh y} + \dfrac{\sinh x \sinh y}{\cosh x \cosh y}}$

$$= \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y}$$

15. Putting $y = x$ in the result from Exercise 11, we have

$$\sinh 2x = \sinh(x+x) = \sinh x \cosh x + \cosh x \sinh x = 2 \sinh x \cosh x.$$

16. Putting $y = x$ in the result from Exercise 12, we have

$$\cosh 2x = \cosh(x+x) = \cosh x \cosh x + \sinh x \sinh x = \cosh^2 x + \sinh^2 x.$$

17. $\tanh(\ln x) = \dfrac{\sinh(\ln x)}{\cosh(\ln x)} = \dfrac{\left(e^{\ln x} - e^{-\ln x}\right)/2}{\left(e^{\ln x} + e^{-\ln x}\right)/2} = \dfrac{x - \left(e^{\ln x}\right)^{-1}}{x + \left(e^{\ln x}\right)^{-1}} = \dfrac{x - x^{-1}}{x + x^{-1}}$

$$= \frac{x - 1/x}{x + 1/x} = \frac{\left(x^2 - 1\right)/x}{\left(x^2 + 1\right)/x} = \frac{x^2 - 1}{x^2 + 1}$$

18. $\dfrac{1 + \tanh x}{1 - \tanh x} = \dfrac{1 + (\sinh x)/\cosh x}{1 - (\sinh x)/\cosh x} = \dfrac{\cosh x + \sinh x}{\cosh x - \sinh x} = \dfrac{\frac{1}{2}\left(e^x + e^{-x}\right) + \frac{1}{2}\left(e^x - e^{-x}\right)}{\frac{1}{2}\left(e^x + e^{-x}\right) - \frac{1}{2}\left(e^x - e^{-x}\right)}$

$$= \frac{e^x + e^{-x} + e^x - e^{-x}}{e^x + e^{-x} - e^x + e^{-x}} = \frac{2e^x}{2e^{-x}} = e^{2x}$$
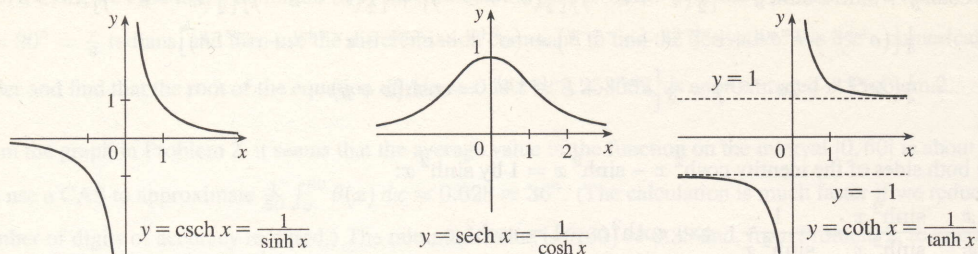
Or: Using the results of Exercises 9 and 10, $\dfrac{\cosh x + \sinh x}{\cosh x - \sinh x} = \dfrac{e^x}{e^{-x}} = e^{2x}$

19. By Exercise 9, $(\cosh x + \sinh x)^n = (e^x)^n = e^{nx} = \cosh nx + \sinh nx$.

20. $\sinh x = \frac{3}{4} \quad \Rightarrow \quad \operatorname{csch} x = 1/\sinh x = \frac{4}{3}$. $\cosh^2 x = \sinh^2 x + 1 = \frac{9}{16} + 1 = \frac{25}{16} \quad \Rightarrow \quad \cosh x = \frac{5}{4}$ (since

$\cosh x > 0$). $\operatorname{sech} x = 1/\cosh x = \frac{4}{5}$, $\tanh x = \sinh x/\cosh x = \frac{3/4}{5/4} = \frac{3}{5}$, and $\coth x = 1/\tanh x = \frac{5}{3}$.

21. $\tanh x = \frac{4}{5} > 0$, so $x > 0$. $\coth x = 1/\tanh x = \frac{5}{4}$, $\operatorname{sech}^2 x = 1 - \tanh^2 x = 1 - \left(\frac{4}{5}\right)^2 = \frac{9}{25} \quad \Rightarrow$

$\operatorname{sech} x = \frac{3}{5}$ (since $\operatorname{sech} x > 0$), $\cosh x = 1/\operatorname{sech} x = \frac{5}{3}$, $\sinh x = \tanh x \cosh x = \frac{4}{5} \cdot \frac{5}{3} = \frac{4}{3}$, and

$\operatorname{csch} x = 1/\sinh x = \frac{3}{4}$.

22.

$$y = \operatorname{csch} x = \frac{1}{\sinh x}$$

$$y = \operatorname{sech} x = \frac{1}{\cosh x}$$

$$y = \coth x = \frac{1}{\tanh x}$$

23. (a) $\displaystyle\lim_{x \to \infty} \tanh x = \lim_{x \to \infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} \cdot \frac{e^{-x}}{e^{-x}} = \lim_{x \to \infty} \frac{1 - e^{-2x}}{1 + e^{-2x}} = \frac{1 - 0}{1 + 0} = 1$

(b) $\displaystyle\lim_{x \to -\infty} \tanh x = \lim_{x \to -\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} \cdot \frac{e^x}{e^x} = \lim_{x \to -\infty} \frac{e^{2x} - 1}{e^{2x} + 1} = \frac{0 - 1}{0 + 1} = -1$

(c) $\displaystyle\lim_{x \to \infty} \sinh x = \lim_{x \to \infty} \frac{e^x - e^{-x}}{2} = \infty$

(d) $\displaystyle\lim_{x \to -\infty} \sinh x = \lim_{x \to -\infty} \frac{e^x - e^{-x}}{2} = -\infty$

(e) $\displaystyle\lim_{x \to \infty} \operatorname{sech} x = \lim_{x \to \infty} \frac{2}{e^x + e^{-x}} = 0$

(f) $\displaystyle\lim_{x \to \infty} \coth x = \lim_{x \to \infty} \frac{e^x + e^{-x}}{e^x - e^{-x}} \cdot \frac{e^{-x}}{e^{-x}} = \lim_{x \to \infty} \frac{1 + e^{-2x}}{1 - e^{-2x}} = \frac{1 + 0}{1 - 0} = 1$ [*Or:* Use part (a)]

(g) $\displaystyle\lim_{x \to 0^+} \coth x = \lim_{x \to 0^+} \frac{\cosh x}{\sinh x} = \infty$, since $\sinh x \to 0$ through positive values and $\cosh x \to 1$.

(h) $\displaystyle\lim_{x \to 0^-} \coth x = \lim_{x \to 0^-} \frac{\cosh x}{\sinh x} = -\infty$, since $\sinh x \to 0$ through negative values and $\cosh x \to 1$.

(i) $\displaystyle\lim_{x \to -\infty} \operatorname{csch} x = \lim_{x \to -\infty} \frac{2}{e^x - e^{-x}} = 0$

24. (a) $\dfrac{d}{dx} \cosh x = \dfrac{d}{dx} \left[\frac{1}{2}(e^x + e^{-x}) \right] = \frac{1}{2}(e^x - e^{-x}) = \sinh x$

(b) $\dfrac{d}{dx} \tanh x = \dfrac{d}{dx} \left[\dfrac{\sinh x}{\cosh x} \right] = \dfrac{\cosh x \cosh x - \sinh x \sinh x}{\cosh^2 x} = \dfrac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} = \dfrac{1}{\cosh^2 x} = \operatorname{sech}^2 x$

(c) $\dfrac{d}{dx} \operatorname{csch} x = \dfrac{d}{dx} \left[\dfrac{1}{\sinh x} \right] = -\dfrac{\cosh x}{\sinh^2 x} = -\dfrac{1}{\sinh x} \cdot \dfrac{\cosh x}{\sinh x} = -\operatorname{csch} x \coth x$

(d) $\dfrac{d}{dx} \operatorname{sech} x = \dfrac{d}{dx} \left[\dfrac{1}{\cosh x} \right] = -\dfrac{\sinh x}{\cosh^2 x} = -\dfrac{1}{\cosh x} \cdot \dfrac{\sinh x}{\cosh x} = -\operatorname{sech} x \tanh x$

(e) $\dfrac{d}{dx} \coth x = \dfrac{d}{dx} \left[\dfrac{\cosh x}{\sinh x} \right] = \dfrac{\sinh x \sinh x - \cosh x \cosh x}{\sinh^2 x} = \dfrac{\sinh^2 x - \cosh^2 x}{\sinh^2 x} = -\dfrac{1}{\sinh^2 x}$

$= -\operatorname{csch}^2 x$

25. Let $y = \sinh^{-1} x$. Then $\sinh y = x$ and, by Example 1(a), $\cosh^2 y - \sinh^2 y = 1$ $\Rightarrow$ [with $\cosh y > 0$]

$\cosh y = \sqrt{1 + \sinh^2 y} = \sqrt{1 + x^2}$. So by Exercise 9, $e^y = \sinh y + \cosh y = x + \sqrt{1 + x^2}$ $\Rightarrow$

$y = \ln\left(x + \sqrt{1 + x^2}\right)$.

29. (a) Let $y = \cosh^{-1} x$. Then $\cosh y = x$ and $y \geq 0 \Rightarrow \sinh y \dfrac{dy}{dx} = 1 \Rightarrow$

$$\frac{dy}{dx} = \frac{1}{\sinh y} = \frac{1}{\sqrt{\cosh^2 y - 1}} = \frac{1}{\sqrt{x^2 - 1}} \quad \text{(since } \sinh y \geq 0 \text{ for } y \geq 0\text{)}. \quad \textit{Or:} \text{ Use Formula 4.}$$

(b) Let $y = \tanh^{-1} x$. Then $\tanh y = x \Rightarrow \text{sech}^2 y \dfrac{dy}{dx} = 1 \Rightarrow \dfrac{dy}{dx} = \dfrac{1}{\text{sech}^2 y} = \dfrac{1}{1 - \tanh^2 y} = \dfrac{1}{1 - x^2}$.

Or: Use Formula 5.

(c) Let $y = \text{csch}^{-1} x$. Then $\text{csch } y = x \Rightarrow -\text{csch } y \coth y \dfrac{dy}{dx} = 1 \Rightarrow \dfrac{dy}{dx} = -\dfrac{1}{\text{csch } y \coth y}$.

By Exercise 13, $\coth y = \pm\sqrt{\text{csch}^2 y + 1} = \pm\sqrt{x^2 + 1}$. If $x > 0$, then $\coth y > 0$, so $\coth y = \sqrt{x^2 + 1}$. If $x < 0$, then $\coth y < 0$, so $\coth y = -\sqrt{x^2 + 1}$. In either case we have

$$\frac{dy}{dx} = -\frac{1}{\text{csch } y \coth y} = -\frac{1}{|x|\sqrt{x^2 + 1}}.$$

(d) Let $y = \text{sech}^{-1} x$. Then $\text{sech } y = x \Rightarrow -\text{sech } y \tanh y \dfrac{dy}{dx} = 1 \Rightarrow$

$$\frac{dy}{dx} = -\frac{1}{\text{sech } y \tanh y} = -\frac{1}{\text{sech } y \sqrt{1 - \text{sech}^2 y}} = -\frac{1}{x\sqrt{1 - x^2}}. \text{ (Note that } y > 0 \text{ and so } \tanh y > 0.\text{)}$$

(e) Let $y = \coth^{-1} x$. Then $\coth y = x \Rightarrow -\text{csch}^2 y \dfrac{dy}{dx} = 1 \Rightarrow$

$$\frac{dy}{dx} = -\frac{1}{\text{csch}^2 y} = \frac{1}{1 - \coth^2 y} = \frac{1}{1 - x^2} \text{ by Exercise 13.}$$

30. $f(x) = \tanh 4x \Rightarrow f'(x) = 4\,\text{sech}^2 4x$

31. $f(x) = x \cosh x \Rightarrow f'(x) = x(\cosh x)' + (\cosh x)(x)' = x \sinh x + \cosh x$

32. $g(x) = \sinh^2 x \Rightarrow g'(x) = 2 \sinh x \cosh x$

33. $h(x) = \sinh(x^2) \Rightarrow h'(x) = \cosh(x^2) \cdot 2x = 2x \cosh(x^2)$

34. $F(x) = \sinh x \tanh x \Rightarrow F'(x) = \sinh x \,\text{sech}^2 x + \tanh x \cosh x$

35. $G(x) = \dfrac{1 - \cosh x}{1 + \cosh x} \Rightarrow$

$$G'(x) = \frac{(1 + \cosh x)(-\sinh x) - (1 - \cosh x)(\sinh x)}{(1 + \cosh x)^2}$$

$$= \frac{-\sinh x - \sinh x \cosh x - \sinh x + \sinh x \cosh x}{(1 + \cosh x)^2} = \frac{-2 \sinh x}{(1 + \cosh x)^2}$$

36. $f(t) = e^t \,\text{sech } t \Rightarrow f'(t) = e^t(-\text{sech } t \tanh t) + (\text{sech } t)e^t = e^t \,\text{sech } t\,(1 - \tanh t)$

37. $h(t) = \coth\sqrt{1 + t^2} \Rightarrow h'(t) = -\text{csch}^2\sqrt{1 + t^2} \cdot \tfrac{1}{2}(1 + t^2)^{-1/2}(2t) = -\dfrac{t\,\text{csch}^2\sqrt{1 + t^2}}{\sqrt{1 + t^2}}$

38. $f(t) = \ln(\sinh t) \Rightarrow f'(t) = \dfrac{1}{\sinh t}\cosh t = \coth t$

39. $H(t) = \tanh(e^t) \Rightarrow H'(t) = \text{sech}^2(e^t) \cdot e^t = e^t \,\text{sech}^2(e^t)$

40. $y = \sinh(\cosh x) \Rightarrow y' = \cosh(\cosh x) \cdot \sinh x$

41. $y = e^{\cosh 3x} \Rightarrow y' = e^{\cosh 3x} \cdot \sinh 3x \cdot 3 = 3e^{\cosh 3x} \sinh 3x$