54. $y = \left(1 + \frac{a}{x}\right)^{bx} \Rightarrow \ln y = bx \ln\left(1 + \frac{a}{x}\right)$, so

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{b \ln(1 + a/x)}{1/x} \overset{H}{=} \lim_{x\to\infty} \frac{b\left(\frac{1}{1 + a/x}\right)\left(-\frac{a}{x^2}\right)}{-1/x^2} = \lim_{x\to\infty} \frac{ab}{1 + a/x} = ab \Rightarrow$$

$$\lim_{x\to\infty} \left(1 + \frac{a}{x}\right)^{bx} = \lim_{x\to\infty} e^{\ln y} = e^{ab}.$$

55. $y = \left(1 + \frac{3}{x} + \frac{5}{x^2}\right)^x \Rightarrow \ln y = x \ln\left(1 + \frac{3}{x} + \frac{5}{x^2}\right) \Rightarrow$

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln\left(1 + \frac{3}{x} + \frac{5}{x^2}\right)}{1/x} \overset{H}{=} \lim_{x\to\infty} \frac{\left(-\frac{3}{x^2} - \frac{10}{x^3}\right)\Big/\left(1 + \frac{3}{x} + \frac{5}{x^2}\right)}{-1/x^2} = \lim_{x\to\infty} \frac{3 + \frac{10}{x}}{1 + \frac{3}{x} + \frac{5}{x^2}} = 3,$$

so $\lim_{x\to\infty} \left(1 + \frac{3}{x} + \frac{5}{x^2}\right)^x = \lim_{x\to\infty} e^{\ln y} = e^3$.

56. $y = x^{(\ln 2)/(1 + \ln x)} \Rightarrow \ln y = \frac{\ln 2}{1 + \ln x} \ln x \Rightarrow$

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{(\ln 2)(\ln x)}{1 + \ln x} \overset{H}{=} \lim_{x\to\infty} \frac{(\ln 2)(1/x)}{1/x} = \lim_{x\to\infty} \ln 2 = \ln 2,$$

so $\lim_{x\to\infty} x^{(\ln 2)/(1 + \ln x)} = \lim_{x\to\infty} e^{\ln y} = e^{\ln 2} = 2$.

57. $y = x^{1/x} \Rightarrow \ln y = (1/x) \ln x \Rightarrow \lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln x}{x} \overset{H}{=} \lim_{x\to\infty} \frac{1/x}{1} = 0 \Rightarrow$

$\lim_{x\to\infty} x^{1/x} = \lim_{x\to\infty} e^{\ln y} = e^0 = 1$

58. $y = (e^x + x)^{1/x} \Rightarrow \ln y = \frac{1}{x} \ln(e^x + x)$, so

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln(e^x + x)}{x} \overset{H}{=} \lim_{x\to\infty} \frac{e^x + 1}{e^x + x} \overset{H}{=} \lim_{x\to\infty} \frac{e^x}{e^x + 1} \overset{H}{=} \lim_{x\to\infty} \frac{e^x}{e^x} = 1 \Rightarrow$$

$\lim_{x\to\infty} (e^x + x)^{1/x} = \lim_{x\to\infty} e^{\ln y} = e^1 = e$.

59. $y = \left(\frac{x}{x+1}\right)^x \Rightarrow \ln y = x \ln\left(\frac{x}{x+1}\right) \Rightarrow$

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} x \ln\left(\frac{x}{x+1}\right) = \lim_{x\to\infty} \frac{\ln x - \ln(x+1)}{1/x} \overset{H}{=} \lim_{x\to\infty} \frac{1/x - 1/(x+1)}{-1/x^2}$$

$$= \lim_{x\to\infty} \left(-x + \frac{x^2}{x+1}\right) = \lim_{x\to\infty} \frac{-x}{x+1} = -1$$

so $\lim_{x\to\infty} \left(\frac{x}{x+1}\right)^x = \lim_{x\to\infty} e^{\ln y} = e^{-1}$

Or: $\lim_{x\to\infty} \left(\frac{x}{x+1}\right)^x = \lim_{x\to\infty} \left[\left(\frac{x+1}{x}\right)^{-1}\right]^x = \left[\lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^x\right]^{-1} = e^{-1}$

60. $y = (\cos 3x)^{5/x} \Rightarrow \ln y = \frac{5}{x} \ln(\cos 3x) \Rightarrow \lim_{x\to 0} \ln y = 5 \lim_{x\to 0} \frac{\ln(\cos 3x)}{x} \overset{H}{=} 5 \lim_{x\to 0} \frac{-3 \tan 3x}{1} = 0,$

so $\lim_{x\to 0} (\cos 3x)^{5/x} = e^0 = 1$.

61. $y = (\cos x)^{1/x^2} \quad \Rightarrow \quad \ln y = \dfrac{1}{x^2}\ln\cos x \quad \Rightarrow$

$$\lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} \frac{\ln\cos x}{x^2} \overset{\text{H}}{=} \lim_{x\to 0^+} \frac{-\tan x}{2x} \overset{\text{H}}{=} \lim_{x\to 0^+} \frac{-\sec^2 x}{2} = -\frac{1}{2} \quad \Rightarrow$$

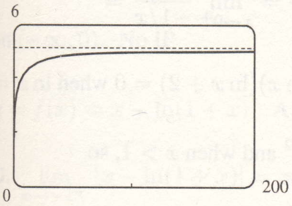
$$\lim_{x\to 0^+} (\cos x)^{1/x^2} = \lim_{x\to 0^+} e^{\ln y} = e^{-1/2} = 1/\sqrt{e}$$

62. $y = \left(\dfrac{2x-3}{2x+5}\right)^{2x+1} \quad \Rightarrow \quad \ln y = (2x+1)\ln\left(\dfrac{2x-3}{2x+5}\right) \quad \Rightarrow$

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln(2x-3) - \ln(2x+5)}{1/(2x+1)} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{2/(2x-3) - 2/(2x+5)}{-2/(2x+1)^2} = \lim_{x\to\infty} \frac{-8(2x+1)^2}{(2x-3)(2x+5)}$$

$$= \lim_{x\to\infty} \frac{-8(2+1/x)^2}{(2-3/x)(2+5/x)} = -8 \quad \Rightarrow \quad \lim_{x\to\infty} \left(\frac{2x-3}{2x+5}\right)^{2x+1} = e^{-8}$$

63.



From the graph, it appears that $\lim_{x\to\infty} x\,[\ln(x+5) - \ln x] = 5$.

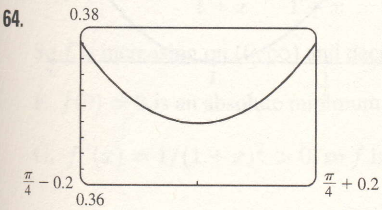
To prove this, we first note that

$\ln(x+5) - \ln x = \ln\dfrac{x+5}{x} = \ln\left(1 + \dfrac{5}{x}\right) \to \ln 1 = 0$ as $x\to\infty$. Thus,

$$\lim_{x\to\infty} x\,[\ln(x+5) - \ln x] = \lim_{x\to\infty} \frac{\ln(x+5) - \ln x}{1/x} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{\dfrac{1}{x+5} - \dfrac{1}{x}}{-1/x^2}$$

$$= \lim_{x\to\infty}\left[\frac{x-(x+5)}{x(x+5)}\cdot\frac{-x^2}{1}\right] = \lim_{x\to\infty}\frac{5x^2}{x^2+5x} = 5$$

64.



From the graph, it appears that $\lim_{x\to\pi/4} (\tan x)^{\tan 2x} \approx 0.368$.

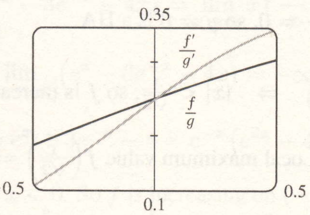
The limit has the form 1^∞. Now $y = (\tan x)^{\tan 2x} \quad \Rightarrow$

$\ln y = \tan 2x\,\ln(\tan x)$, so

$$\lim_{x\to\pi/4} \ln y = \lim_{x\to\pi/4} \frac{\ln(\tan x)}{\cot 2x} \overset{\text{H}}{=} \lim_{x\to\pi/4} \frac{\sec^2 x/\tan x}{-2\csc^2 2x} = \frac{2/1}{-2(1)} = -1$$

$$\Rightarrow \quad \lim_{x\to\pi/4} (\tan x)^{\tan 2x} = \lim_{x\to\pi/4} e^{\ln y} = e^{-1} = 1/e \approx 0.3679.$$

65.

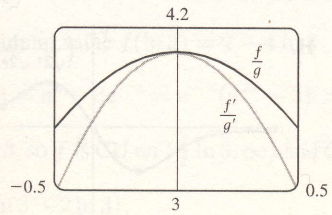


From the graph, it appears that

$\lim_{x\to 0}\dfrac{f(x)}{g(x)} = \lim_{x\to 0}\dfrac{f'(x)}{g'(x)} = 0.25$. We calculate

$$\lim_{x\to 0}\frac{f(x)}{g(x)} = \lim_{x\to 0}\frac{e^x - 1}{x^3 + 4x} \overset{\text{H}}{=} \lim_{x\to 0}\frac{e^x}{3x^2+4} = \frac{1}{4}.$$

66.



From the graph, it appears that $\lim_{x\to 0}\dfrac{f(x)}{g(x)} = \lim_{x\to 0}\dfrac{f'(x)}{g'(x)} = 4$.

We calculate

$$\lim_{x\to 0}\frac{f(x)}{g(x)} = \lim_{x\to 0}\frac{2x\sin x}{\sec x - 1} \overset{\text{H}}{=} \lim_{x\to 0}\frac{2(x\cos x + \sin x)}{\sec x\,\tan x}$$

$$\overset{\text{H}}{=} \lim_{x\to 0}\frac{2(-x\sin x + \cos x + \cos x)}{\sec x(\sec^2 x) + \tan x(\sec x\,\tan x)} = \frac{4}{1} = 4$$