76. (a) The primes less than 25 are 2, 3, 5, 7, 11, 13, 17, 19, and 23. There are 9 of them, so $\pi(25) = 9$. We use the sieve of Eratosthenes, and arrive at the figure at right. There are 25 numbers left over, so $\pi(100) = 25$.

	2	3	~~4~~	5	~~6~~	7	~~8~~	~~9~~	~~10~~
11	~~12~~	13	~~14~~	~~15~~	~~16~~	17	~~18~~	19	~~20~~
~~21~~	~~22~~	23	~~24~~	~~25~~	~~26~~	~~27~~	~~28~~	29	~~30~~
31	~~32~~	~~33~~	~~34~~	~~35~~	~~36~~	37	~~38~~	~~39~~	~~40~~
41	~~42~~	43	~~44~~	~~45~~	~~46~~	47	~~48~~	~~49~~	~~50~~
~~51~~	~~52~~	53	~~54~~	~~55~~	~~56~~	~~57~~	~~58~~	59	~~60~~
61	~~62~~	~~63~~	~~64~~	~~65~~	~~66~~	67	~~68~~	~~69~~	~~70~~
71	~~72~~	73	~~74~~	~~75~~	~~76~~	~~77~~	~~78~~	79	~~80~~
~~81~~	~~82~~	83	~~84~~	~~85~~	~~86~~	~~87~~	~~88~~	89	~~90~~
91	~~92~~	~~93~~	~~94~~	~~95~~	~~96~~	97	~~98~~	~~99~~	~~100~~

(b) Let $f(n) = \dfrac{\pi(n)}{n/\ln n}$. We compute $f(100) = \dfrac{25}{100/\ln 100} \approx 1.15$, $f(1000) \approx 1.16$, $f(10^4) \approx 1.13$, $f(10^5) \approx 1.10$, $f(10^6) \approx 1.08$, and $f(10^7) \approx 1.07$.

(c) By the Prime Number Theorem, the number of primes less than a billion, that is, $\pi(10^9)$, should be close to $10^9/\ln 10^9 \approx 48{,}254{,}942$. In fact, $\pi(10^9) = 50{,}847{,}543$, so our estimate is off by about 5.1%. Do not attempt this calculation at home.

7.4 Derivatives of Logarithmic Functions

1. The differentiation formula for logarithmic functions, $\dfrac{d}{dx}(\log_a x) = \dfrac{1}{x\ln a}$, is simplest when $a = e$ because $\ln e = 1$.

2. $f(x) = \ln(x^2+10) \;\Rightarrow\; f'(x) = \dfrac{1}{x^2+10}\dfrac{d}{dx}(x^2+10) = \dfrac{2x}{x^2+10}$

3. $f(\theta) = \ln(\cos\theta) \;\Rightarrow\; f'(\theta) = \dfrac{1}{\cos\theta}\dfrac{d}{d\theta}(\cos\theta) = \dfrac{-\sin\theta}{\cos\theta} = -\tan\theta$

4. $f(x) = \cos(\ln x) \;\Rightarrow\; f'(x) = -\sin(\ln x)\cdot\dfrac{1}{x} = \dfrac{-\sin(\ln x)}{x}$

5. $f(x) = \log_2(1-3x) \;\Rightarrow\; f'(x) = \dfrac{1}{(1-3x)\ln 2}\dfrac{d}{dx}(1-3x) = \dfrac{-3}{(1-3x)\ln 2}$ or $\dfrac{3}{(3x-1)\ln 2}$

6. $f(x) = \log_{10}\left(\dfrac{x}{x-1}\right) = \log_{10} x - \log_{10}(x-1) \;\Rightarrow\; f'(x) = \dfrac{1}{x\ln 10} - \dfrac{1}{(x-1)\ln 10}$ or $-\dfrac{1}{x(x-1)\ln 10}$

7. $f(x) = \sqrt[5]{\ln x} = (\ln x)^{1/5} \;\Rightarrow\; f'(x) = \frac{1}{5}(\ln x)^{-4/5}\dfrac{d}{dx}(\ln x) = \dfrac{1}{5(\ln x)^{4/5}}\cdot\dfrac{1}{x} = \dfrac{1}{5x\sqrt[5]{(\ln x)^4}}$

8. $f(x) = \ln\sqrt[5]{x} = \ln x^{1/5} = \frac{1}{5}\ln x \;\Rightarrow\; f'(x) = \dfrac{1}{5}\cdot\dfrac{1}{x} = \dfrac{1}{5x}$

9. $f(x) = \sqrt{x}\ln x \;\Rightarrow\; f'(x) = \sqrt{x}\left(\dfrac{1}{x}\right) + (\ln x)\cdot\dfrac{1}{2\sqrt{x}} = \dfrac{1}{\sqrt{x}} + \dfrac{\ln x}{2\sqrt{x}} = \dfrac{2+\ln x}{2\sqrt{x}}$

10. $f(t) = \dfrac{1+\ln t}{1-\ln t} \;\Rightarrow$

$$f'(t) = \frac{(1-\ln t)(1/t) - (1+\ln t)(-1/t)}{(1-\ln t)^2} = \frac{(1/t)[(1-\ln t)+(1+\ln t)]}{(1-\ln t)^2} = \frac{2}{t(1-\ln t)^2}$$

11. $F(t) = \ln\dfrac{(2t+1)^3}{(3t-1)^4} = \ln(2t+1)^3 - \ln(3t-1)^4 = 3\ln(2t+1) - 4\ln(3t-1) \;\Rightarrow$

$$F'(t) = 3\cdot\frac{1}{2t+1}\cdot 2 - 4\cdot\frac{1}{3t-1}\cdot 3 = \frac{6}{2t+1} - \frac{12}{3t-1}, \text{ or combined, } \frac{-6(t+3)}{(2t+1)(3t-1)}.$$

12. $h(x) = \ln\left(x + \sqrt{x^2 - 1}\right) \Rightarrow$

$$h'(x) = \frac{1}{x + \sqrt{x^2 - 1}}\left(1 + \frac{x}{\sqrt{x^2 - 1}}\right) = \frac{1}{x + \sqrt{x^2 - 1}} \cdot \frac{\sqrt{x^2 - 1} + x}{\sqrt{x^2 - 1}} = \frac{1}{\sqrt{x^2 - 1}}$$

13. $g(x) = \ln \dfrac{a - x}{a + x} = \ln(a - x) - \ln(a + x) \Rightarrow$

$$g'(x) = \frac{1}{a - x}(-1) - \frac{1}{a + x} = \frac{-(a + x) - (a - x)}{(a - x)(a + x)} = \frac{-2a}{a^2 - x^2}$$

14. $F(y) = y\ln(1 + e^y) \Rightarrow F'(y) = y \cdot \dfrac{1}{1 + e^y} \cdot e^y + \ln(1 + e^y) \cdot 1 = \dfrac{ye^y}{1 + e^y} + \ln(1 + e^y)$

15. $f(u) = \dfrac{\ln u}{1 + \ln(2u)} \Rightarrow$

$$f'(u) = \frac{[1 + \ln(2u)] \cdot \frac{1}{u} - \ln u \cdot \frac{1}{2u} \cdot 2}{[1 + \ln(2u)]^2} = \frac{\frac{1}{u}\,[1 + \ln(2u) - \ln u]}{[1 + \ln(2u)]^2}$$

$$= \frac{1 + (\ln 2 + \ln u) - \ln u}{u\,[1 + \ln(2u)]^2} = \frac{1 + \ln 2}{u\,[1 + \ln(2u)]^2}$$

16. $y = \ln(x^4 \sin^2 x) = \ln x^4 + \ln(\sin x)^2 = 4\ln x + 2\ln\sin x \Rightarrow y' = 4 \cdot \dfrac{1}{x} + 2 \cdot \dfrac{1}{\sin x} \cdot \cos x = \dfrac{4}{x} + 2\cot x$

17. $h(t) = t^3 - 3^t \Rightarrow h'(t) = 3t^2 - 3^t \ln 3$

18. $y = 10^{\tan\theta} \Rightarrow y' = 10^{\tan\theta}(\ln 10)(\sec^2\theta)$

19. $y = \ln\left|2 - x - 5x^2\right| \Rightarrow y' = \dfrac{1}{2 - x - 5x^2} \cdot (-1 - 10x) = \dfrac{-10x - 1}{2 - x - 5x^2}$ or $\dfrac{10x + 1}{5x^2 + x - 2}$

20. $G(u) = \ln\sqrt{\dfrac{3u + 2}{3u - 2}} = \frac{1}{2}[\ln(3u + 2) - \ln(3u - 2)] \Rightarrow G'(u) = \dfrac{1}{2}\left(\dfrac{3}{3u + 2} - \dfrac{3}{3u - 2}\right) = \dfrac{-6}{9u^2 - 4}$

21. $y = \ln(e^{-x} + xe^{-x}) = \ln(e^{-x}(1 + x)) = \ln(e^{-x}) + \ln(1 + x) = -x + \ln(1 + x) \Rightarrow$

$$y' = -1 + \frac{1}{1 + x} = \frac{-1 - x + 1}{1 + x} = -\frac{x}{1 + x}$$

22. $y = [\ln(1 + e^x)]^2 \Rightarrow y' = 2\,[\ln(1 + e^x)] \cdot \dfrac{1}{1 + e^x} \cdot e^x = \dfrac{2e^x \ln(1 + e^x)}{1 + e^x}$

23. Using Formula 7 and the Chain Rule, $y = 5^{-1/x} \Rightarrow y' = 5^{-1/x}(\ln 5)[-1 \cdot (-x^{-2})] = 5^{-1/x}(\ln 5)/x^2$

24. $y = 2^{3^{x^2}} \Rightarrow y' = 2^{3^{x^2}}(\ln 2)\dfrac{d}{dx}\left(3^{x^2}\right) = 2^{3^{x^2}}(\ln 2)3^{x^2}(\ln 3)(2x)$

25. $y = x\ln x \Rightarrow y' = x(1/x) + (\ln x) \cdot 1 = 1 + \ln x \Rightarrow y'' = 1/x$

26. $y = \dfrac{\ln x}{x^2} \Rightarrow y' = \dfrac{x^2(1/x) - (\ln x)(2x)}{(x^2)^2} = \dfrac{x(1 - 2\ln x)}{x^4} = \dfrac{1 - 2\ln x}{x^3} \Rightarrow$

$$y'' = \frac{x^3(-2/x) - (1 - 2\ln x)(3x^2)}{(x^3)^2} = \frac{x^2(-2 - 3 + 6\ln x)}{x^6} = \frac{6\ln x - 5}{x^4}$$

27. $y = \log_{10} x \Rightarrow y' = \dfrac{1}{x\ln 10} = \dfrac{1}{\ln 10}\left(\dfrac{1}{x}\right) \Rightarrow y'' = \dfrac{1}{\ln 10}\left(-\dfrac{1}{x^2}\right) = -\dfrac{1}{x^2\ln 10}$

28. $y = \ln(\sec x + \tan x) \Rightarrow y' = \dfrac{\sec x\,\tan x + \sec^2 x}{\sec x + \tan x} = \sec x \Rightarrow y'' = \sec x\,\tan x$

29. $f(x) = \dfrac{x}{1 - \ln(x-1)} \Rightarrow$

$$f'(x) = \frac{[1-\ln(x-1)]\cdot 1 - x\cdot\dfrac{-1}{x-1}}{[1-\ln(x-1)]^2} = \frac{\dfrac{(x-1)[1-\ln(x-1)]+x}{x-1}}{[1-\ln(x-1)]^2} = \frac{x-1-(x-1)\ln(x-1)+x}{(x-1)[1-\ln(x-1)]^2}$$
$$= \frac{2x-1-(x-1)\ln(x-1)}{(x-1)[1-\ln(x-1)]^2}$$

$$\text{Dom}(f) = \{x \mid x-1>0 \text{ and } 1-\ln(x-1)\neq 0\} = \{x \mid x>1 \text{ and } \ln(x-1)\neq 1\}$$
$$= \{x \mid x>1 \text{ and } x-1\neq e^1\} = \{x \mid x>1 \text{ and } x\neq 1+e\} = (1, 1+e)\cup(1+e,\infty)$$

30. $f(x) = \dfrac{1}{1+\ln x} \Rightarrow f'(x) = -\dfrac{1/x}{(1+\ln x)^2}$ [Reciprocal Rule] $= -\dfrac{1}{x(1+\ln x)^2}$.

$\text{Dom}(f) = \{x \mid x>0 \text{ and } \ln x\neq -1\} = \{x \mid x>0 \text{ and } x\neq 1/e\} = (0,1/e)\cup(1/e,\infty)$.

31. $f(x) = x^2\ln(1-x^2) \Rightarrow f'(x) = 2x\ln(1-x^2) + \dfrac{x^2(-2x)}{1-x^2} = 2x\ln(1-x^2) - \dfrac{2x^3}{1-x^2}$.

$\text{Dom}(f) = \{x \mid 1-x^2>0\} = \{x \mid |x|<1\} = (-1,1)$.

32. $f(x) = \ln\ln\ln x \Rightarrow f'(x) = \dfrac{1}{\ln\ln x}\cdot\dfrac{1}{\ln x}\cdot\dfrac{1}{x}$.

$\text{Dom}(f) = \{x \mid \ln\ln x>0\} = \{x \mid \ln x>1\} = \{x \mid x>e\} = (e,\infty)$.

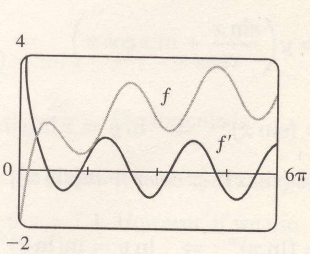
33. $f(x) = \dfrac{x}{\ln x} \Rightarrow f'(x) = \dfrac{\ln x - x(1/x)}{(\ln x)^2} = \dfrac{\ln x - 1}{(\ln x)^2} \Rightarrow f'(e) = \dfrac{1-1}{1^2} = 0$

34. $f(x) = x^2\ln x \Rightarrow f'(x) = 2x\ln x + x^2\left(\dfrac{1}{x}\right) = 2x\ln x + x \Rightarrow f'(1) = 2\ln 1 + 1 = 1$

35. $y = f(x) = \ln\ln x \Rightarrow f'(x) = \dfrac{1}{\ln x}\left(\dfrac{1}{x}\right) \Rightarrow f'(e) = \dfrac{1}{e}$, so an equation of the tangent line at $(e,0)$ is $y - 0 = \dfrac{1}{e}(x-e)$, or $y = \dfrac{1}{e}x - 1$, or $x - ey = e$.

36. $y = \ln(x^3-7) \Rightarrow y' = \dfrac{1}{x^3-7}\cdot 3x^2 \Rightarrow y'(2) = \dfrac{12}{8-7} = 12$, so an equation of a tangent line at $(2,0)$ is $y - 0 = 12(x-2)$ or $y = 12x - 24$.

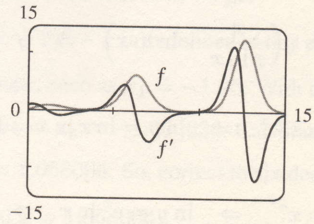
37. $f(x) = \sin x + \ln x \Rightarrow f'(x) = \cos x + 1/x$. This is reasonable, because the graph shows that f increases when f' is positive, and $f'(x) = 0$ when f has a horizontal tangent.



38. $f(x) = x^{\cos x} = e^{\ln x\cos x} \Rightarrow$

$$f'(x) = e^{\ln x\cos x}\left[\ln x(-\sin x) + \cos x\left(\frac{1}{x}\right)\right]$$
$$= x^{\cos x}\left[\frac{\cos x}{x} - \sin x\ln x\right]$$

This is reasonable, because the graph shows that f increases when $f'(x)$ is positive.



39. $y = (2x+1)^5(x^4-3)^6 \Rightarrow \ln y = \ln\left((2x+1)^5(x^4-3)^6\right) \Rightarrow$

$\ln y = 5\ln(2x+1) + 6\ln(x^4-3) \Rightarrow \frac{1}{y}y' = 5\cdot\frac{1}{2x+1}\cdot 2 + 6\cdot\frac{1}{x^4-3}\cdot 4x^3 \Rightarrow$

$y' = y\left(\frac{10}{2x+1} + \frac{24x^3}{x^4-3}\right) = (2x+1)^5(x^4-3)^6\left(\frac{10}{2x+1} + \frac{24x^3}{x^4-3}\right).$

[The answer could be simplified to $y' = 2(2x+1)^4(x^4-3)^5(29x^4+12x^3-15)$, but this is unnecessary.]

40. $y = \sqrt{x}\,e^{x^2}(x^2+1)^{10} \Rightarrow \ln y = \ln\sqrt{x} + \ln e^{x^2} + \ln(x^2+1)^{10} \Rightarrow \ln y = \frac{1}{2}\ln x + x^2 + 10\ln(x^2+1)$

$\Rightarrow \frac{1}{y}y' = \frac{1}{2}\cdot\frac{1}{x} + 2x + 10\cdot\frac{1}{x^2+1}\cdot 2x \Rightarrow y' = \sqrt{x}\,e^{x^2}(x^2+1)^{10}\left(\frac{1}{2x} + 2x + \frac{20x}{x^2+1}\right)$

41. $y = \frac{\sin^2 x\,\tan^4 x}{(x^2+1)^2} \Rightarrow \ln y = \ln(\sin^2 x\,\tan^4 x) - \ln(x^2+1)^2 \Rightarrow$

$\ln y = \ln(\sin x)^2 + \ln(\tan x)^4 - \ln(x^2+1)^2 \Rightarrow \ln y = 2\ln|\sin x| + 4\ln|\tan x| - 2\ln(x^2+1) \Rightarrow$

$\frac{1}{y}y' = 2\cdot\frac{1}{\sin x}\cdot\cos x + 4\cdot\frac{1}{\tan x}\cdot\sec^2 x - 2\cdot\frac{1}{x^2+1}\cdot 2x \Rightarrow$

$y' = \frac{\sin^2 x\,\tan^4 x}{(x^2+1)^2}\left(2\cot x + \frac{4\sec^2 x}{\tan x} - \frac{4x}{x^2+1}\right)$

42. $y = \sqrt[4]{\frac{x^2+1}{x^2-1}} \Rightarrow \ln y = \frac{1}{4}\ln(x^2+1) - \frac{1}{4}\ln(x^2-1) \Rightarrow \frac{1}{y}y' = \frac{1}{4}\cdot\frac{1}{x^2+1}\cdot 2x - \frac{1}{4}\cdot\frac{1}{x^2-1}\cdot 2x \Rightarrow$

$y' = \sqrt[4]{\frac{x^2+1}{x^2-1}}\cdot\frac{1}{2}\left(\frac{x}{x^2+1} - \frac{x}{x^2-1}\right) = \frac{1}{2}\sqrt[4]{\frac{x^2+1}{x^2-1}}\left(\frac{-2x}{x^4-1}\right) = \frac{x}{1-x^4}\sqrt[4]{\frac{x^2+1}{x^2-1}}$

43. $y = x^x \Rightarrow \ln y = \ln x^x \Rightarrow \ln y = x\ln x \Rightarrow y'/y = x(1/x) + (\ln x)\cdot 1 \Rightarrow$

$y' = y(1+\ln x) \Rightarrow y' = x^x(1+\ln x)$

44. $y = x^{1/x} \Rightarrow \ln y = \frac{1}{x}\ln x \Rightarrow \frac{y'}{y} = \frac{1}{x}\left(\frac{1}{x}\right) + (\ln x)\left(-\frac{1}{x^2}\right) \Rightarrow y' = x^{1/x}\frac{1-\ln x}{x^2}$

45. $y = x^{\sin x} \Rightarrow \ln y = \ln x^{\sin x} \Rightarrow \ln y = \sin x\ln x \Rightarrow \frac{y'}{y} = (\sin x)\cdot\frac{1}{x} + (\ln x)(\cos x) \Rightarrow$

$y' = y\left(\frac{\sin x}{x} + \ln x\cos x\right) \Rightarrow y' = x^{\sin x}\left(\frac{\sin x}{x} + \ln x\cos x\right)$

46. $y = (\sin x)^x \Rightarrow \ln y = x\ln(\sin x) \Rightarrow \frac{y'}{y} = x\cdot\frac{1}{\sin x}\cdot\cos x + [\ln(\sin x)]\cdot 1 \Rightarrow$

$y' = (\sin x)^x\,[x\cot x + \ln(\sin x)]$

47. $y = (\ln x)^x \Rightarrow \ln y = \ln(\ln x)^x \Rightarrow \ln y = x\ln\ln x \Rightarrow \frac{y'}{y} = x\cdot\frac{1}{\ln x}\cdot\frac{1}{x} + (\ln\ln x)\cdot 1 \Rightarrow$

$y' = y\left(\frac{x}{x\ln x} + \ln\ln x\right) \Rightarrow y' = (\ln x)^x\left(\frac{1}{\ln x} + \ln\ln x\right)$

48. $y = x^{\ln x} \Rightarrow \ln y = \ln x\ln x = (\ln x)^2 \Rightarrow \frac{y'}{y} = 2\ln x\left(\frac{1}{x}\right) \Rightarrow y' = x^{\ln x}\left(\frac{2\ln x}{x}\right)$

49. $y = x^{e^x} \Rightarrow \ln y = e^x\ln x \Rightarrow \frac{y'}{y} = e^x\cdot\frac{1}{x} + (\ln x)\cdot e^x \Rightarrow y' = x^{e^x}e^x\left(\ln x + \frac{1}{x}\right)$