87. (a) By Exercise 85(a), the result holds for $n = 1$. Suppose that $e^x \geq 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^k}{k!}$ for $x \geq 0$.

Let $f(x) = e^x - 1 - x - \frac{x^2}{2!} - \cdots - \frac{x^k}{k!} - \frac{x^{k+1}}{(k+1)!}$. Then $f'(x) = e^x - 1 - x - \cdots - \frac{x^k}{k!} \geq 0$ by assumption. Hence $f(x)$ is increasing on $(0, \infty)$. So $0 \leq x$ implies that $0 = f(0) \leq f(x) = e^x - 1 - x - \cdots - \frac{x^k}{k!} - \frac{x^{k+1}}{(k+1)!}$, and hence $e^x \geq 1 + x + \cdots + \frac{x^k}{k!} + \frac{x^{k+1}}{(k+1)!}$ for $x \geq 0$. Therefore, for $x \geq 0$, $e^x \geq 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$ for every positive integer n, by mathematical induction.

(b) Taking $n = 4$ and $x = 1$ in (a), we have $e = e^1 \geq 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} = 2.708\overline{3} > 2.7$.

(c) $e^x \geq 1 + x + \cdots + \frac{x^k}{k!} + \frac{x^{k+1}}{(k+1)!} \quad \Rightarrow \quad \frac{e^x}{x^k} \geq \frac{1}{x^k} + \frac{1}{x^{k-1}} + \cdots + \frac{1}{k!} + \frac{x}{(k+1)!} \geq \frac{x}{(k+1)!}$.

But $\lim\limits_{x \to \infty} \frac{x}{(k+1)!} = \infty$, so $\lim\limits_{x \to \infty} \frac{e^x}{x^k} = \infty$.

7.3 Logarithmic Functions

1. (a) It is defined as the inverse of the exponential function with base a, that is, $\log_a x = y \quad \Leftrightarrow \quad a^y = x$.

(b) $(0, \infty)$ (c) $\mathbb{R}$ (d) See Figure 1.

2. (a) The natural logarithm is the logarithm with base e, denoted $\ln x$.

(b) The common logarithm is the logarithm with base 10, denoted $\log x$.

(c) See Figure 3.

3. (a) $\log_{10} 1000 = 3$ because $10^3 = 1000$. *Or:* $\log_{10} 1000 = \log_{10} 10^3 = 3$ by (2).

(b) $\log_{16} 4 = \frac{1}{2}$ because $16^{1/2} = 4$. *Or:* $\log_{16} 4 = \log_{16} 16^{1/2} = \frac{1}{2}$ by (2).

4. (a) By (6), $\ln e^{-100} = -100$. (b) $\log_3 81 = 4$ since $3^4 = 81$.

5. (a) $\log_5 \frac{1}{25} = \log_5 5^{-2} = -2$ by (2). (b) $e^{\ln 15} = 15$ by (6).

6. (a) $\log_{10} 0.1 = -1$ since $10^{-1} = 0.1$.

(b) $\log_8 320 - \log_8 5 = \log_8 \frac{320}{5} = \log_8 64 = 2$ since $8^2 = 64$.

7. (a) $\log_{12} 3 + \log_{12} 48 = \log_{12}(3 \cdot 48) = \log_{12} 144 = 2$ since $12^2 = 144$.

(b) $\log_2 5 - \log_2 90 + 2\log_2 3 = \log_2 5 + \log_2 3^2 - \log_2 90 = \log_2(5 \cdot 9) - \log_2 90$

$= \log_2\left(\frac{45}{90}\right) = \log_2\left(\frac{1}{2}\right) = -1$ since $2^{-1} = \frac{1}{2}$.

8. (a) $2^{(\log_2 3 + \log_2 5)} = 2^{\log_2 15} = 15$ $\quad$ [*Or:* $2^{(\log_2 3 + \log_2 5)} = 2^{\log_2 3} \cdot 2^{\log_2 5} = 3 \cdot 5 = 15$]

(b) $e^{3\ln 2} = e^{\ln(2^3)} = e^{\ln 8} = 8$ $\quad$ [*Or:* $e^{3\ln 2} = \left(e^{\ln 2}\right)^3 = 2^3 = 8$]

9. $\log_2\left(\frac{x^3 y}{z^2}\right) = \log_2(x^3 y) - \log_2 z^2 = \log_2 x^3 + \log_2 y - \log_2 z^2 = 3\log_2 x + \log_2 y - 2\log_2 z$

(assuming that the variables are positive)

10. $\ln\sqrt{a(b^2 + c^2)} = \ln(a(b^2 + c^2))^{1/2} = \frac{1}{2}\ln(a(b^2 + c^2)) = \frac{1}{2}[\ln a + \ln(b^2 + c^2)]$

$= \frac{1}{2}\ln a + \frac{1}{2}\ln(b^2 + c^2)$

11. $\ln(uv)^{10} = 10\ln(uv) = 10(\ln u + \ln v) = 10\ln u + 10\ln v$

12. $\ln \dfrac{3x^2}{(x+1)^5} = \ln 3x^2 - \ln(x+1)^5 = \ln 3 + \ln x^2 - 5\ln(x+1) = \ln 3 + 2\ln x - 5\ln(x+1)$

13. $\log_{10} a - \log_{10} b + \log_{10} c = \log_{10} \dfrac{a}{b} + \log_{10} c = \log_{10}\left(\dfrac{a}{b}\cdot c\right) = \log_{10}\dfrac{ac}{b}$

14. $\ln(x+y) + \ln(x-y) - 2\ln z = \ln((x+y)(x-y)) - \ln z^2 = \ln(x^2-y^2) - \ln z^2 = \ln\dfrac{x^2-y^2}{z^2}$

15. $2\ln 4 - \ln 2 = \ln 4^2 - \ln 2 = \ln 16 - \ln 2 = \ln\frac{16}{2} = \ln 8$

16. $\ln 3 + \frac{1}{3}\ln 8 = \ln 3 + \ln 8^{1/3} = \ln 3 + \ln 2 = \ln(3\cdot 2) = \ln 6$

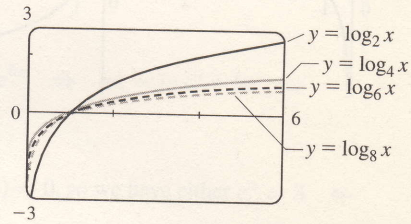
17. $\frac{1}{2}\ln x - 5\ln(x^2+1) = \ln x^{1/2} - \ln(x^2+1)^5 = \ln\dfrac{\sqrt{x}}{(x^2+1)^5}$

18. $\ln x + a\ln y - b\ln z = \ln x + \ln y^a - \ln z^b = \ln(x\cdot y^a) - \ln z^b = \ln\left(xy^a/z^b\right)$

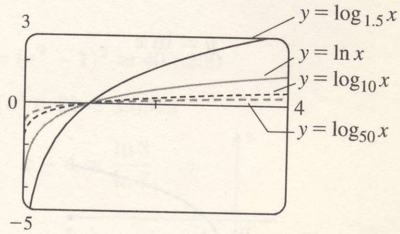
19. (a) $\log_{12} e = \dfrac{\ln e}{\ln 12} = \dfrac{1}{\ln 12} \approx 0.402430$ (b) $\log_6 13.54 = \dfrac{\ln 13.54}{\ln 6} \approx 1.454240$

(c) $\log_2 \pi = \dfrac{\ln \pi}{\ln 2} \approx 1.651496$

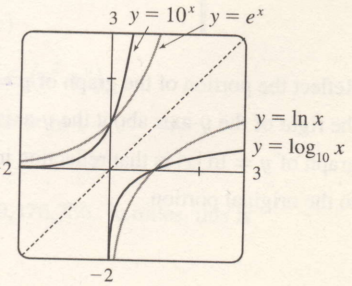
20. To graph the functions, we use $\log_2 x = \dfrac{\ln x}{\ln 2}$, $\log_4 x = \dfrac{\ln x}{\ln 4}$, etc. These graphs all approach $-\infty$ as $x \to 0^+$, and they all pass through the point $(1, 0)$. Also, they are all increasing, and all approach ∞ as $x \to \infty$. The smaller the base, the larger the rate of increase of the function (for $x > 1$) and the closer the approach to the y-axis (as $x \to 0^+$).



21. To graph these functions, we use $\log_{1.5} x = \dfrac{\ln x}{\ln 1.5}$ and $\log_{50} x = \dfrac{\ln x}{\ln 50}$. These graphs all approach $-\infty$ as $x \to 0^+$, and they all pass through the point $(1, 0)$. Also, they are all increasing, and all approach ∞ as $x \to \infty$. The functions with larger bases increase extremely slowly, and the ones with smaller bases do so somewhat more quickly. The functions with large bases approach the y-axis more closely as $x \to 0^+$.

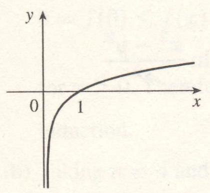


22. We see that the graph of $\ln x$ is the reflection of the graph of e^x about the line $y = x$, and that the graph of $\log_{10} x$ is the reflection of the graph of 10^x about the same line. The graph of 10^x increases more quickly than that of e^x. Also note that $\log_{10} x \to \infty$ as $x \to \infty$ more slowly than $\ln x$.

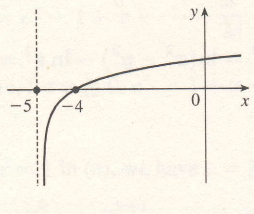


23. Shift the graph of $y = \log_{10} x$ five units to the left to obtain the graph of $y = \log_{10}(x + 5)$. Note the vertical asymptote of $x = -5$.

$y = \log_{10} x$

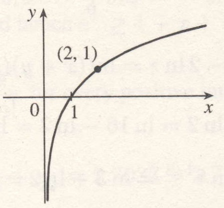


$y = \log_{10}(x + 5)$

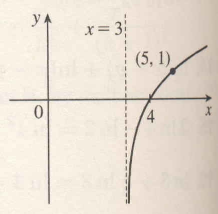


24. $\log_2(x - 3)$: Start with the graph of $y = \log_2 x$ and shift 3 units to the right.

$y = \log_2 x$

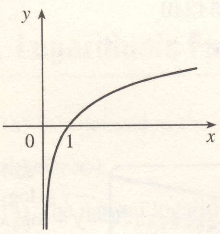


$y = \log_2(x - 3)$

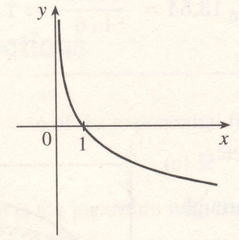


25. Reflect the graph of $y = \ln x$ about the x-axis to obtain the graph of $y = -\ln x$.

$y = \ln x$

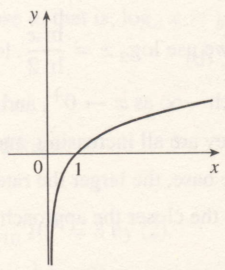


$y = -\ln x$

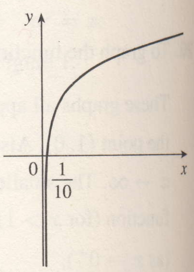


26. $y = \ln(10x)$: Start with the graph of $y = \ln x$ and compress horizontally by a factor of 10. *Or*: $y = \ln(10x) = \ln 10 + \ln x$, so we could start with $y = \ln x$ and shift ln 10 units upward.

$y = \ln x$

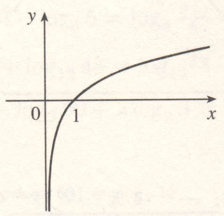


$y = \ln(10x)$

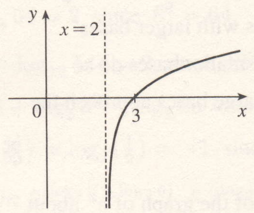


27. $y = 5 + \ln(x - 2)$: Start with the graph of $y = \ln x$, shift 2 units to the right and then shift 5 units upward.

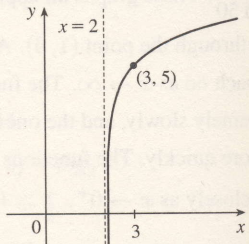
$y = \ln x$



$y = \ln(x - 2)$

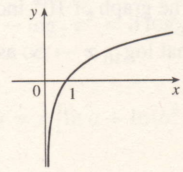


$y = 5 + \ln(x - 2)$

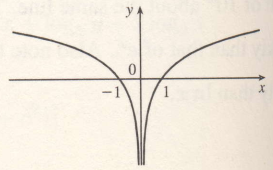


28. Reflect the portion of the graph of $y = \ln x$ to the right of the y-axis about the y-axis. The graph of $y = \ln |x|$ is that reflection in addition to the original portion.

$y = \ln x$



$y = \ln |x|$



29. (a) $2\ln x = 1 \quad\Rightarrow\quad \ln x = \frac{1}{2} \quad\Rightarrow\quad x = e^{1/2} = \sqrt{e}$

(b) $e^{-x} = 5 \quad\Rightarrow\quad -x = \ln 5 \quad\Rightarrow\quad x = -\ln 5$

30. (a) $e^{2x+3} - 7 = 0 \quad\Rightarrow\quad e^{2x+3} = 7 \quad\Rightarrow\quad 2x+3 = \ln 7 \quad\Rightarrow\quad 2x = \ln 7 - 3 \quad\Rightarrow\quad x = \frac{1}{2}(\ln 7 - 3)$

(b) $\ln(5-2x) = -3 \quad\Rightarrow\quad 5 - 2x = e^{-3} \quad\Rightarrow\quad 2x = 5 - e^{-3} \quad\Rightarrow\quad x = \frac{1}{2}(5 - e^{-3})$

31. (a) $5^{x-3} = 10 \quad\Leftrightarrow\quad \log_{10} 5^{x-3} = \log_{10} 10 \quad\Leftrightarrow\quad (x-3)\log_{10} 5 = 1 \quad\Leftrightarrow\quad x - 3 = 1/\log_{10} 5 \quad\Leftrightarrow$
$x = 3 + 1/\log_{10} 5$

(b) $\log_{10}(x+1) = 4 \quad\Leftrightarrow\quad x + 1 = 10^4 \quad\Leftrightarrow\quad x = 10{,}000 - 1 = 9999$

32. (a) $e^{3x+1} = k \quad\Leftrightarrow\quad 3x + 1 = \ln k \quad\Leftrightarrow\quad x = \frac{1}{3}(\ln k - 1)$

(b) $\log_2(mx) = c \quad\Leftrightarrow\quad mx = 2^c \quad\Leftrightarrow\quad x = 2^c/m$

33. $\ln(\ln x) = 1 \quad\Leftrightarrow\quad e^{\ln(\ln x)} = e^1 \quad\Leftrightarrow\quad \ln x = e^1 = e \quad\Leftrightarrow\quad e^{\ln x} = e^e \quad\Leftrightarrow\quad x = e^e$

34. $e^{e^x} = 10 \quad\Leftrightarrow\quad \ln\left(e^{e^x}\right) = \ln 10 \quad\Leftrightarrow\quad e^x \ln e = e^x = \ln 10 \quad\Leftrightarrow\quad \ln e^x = \ln(\ln 10) \quad\Leftrightarrow\quad x = \ln(\ln 10)$

35. $2\ln x = \ln 2 + \ln(3x-4) \quad\Rightarrow\quad \ln x^2 = \ln[2(3x-4)] \quad\Rightarrow\quad \ln x^2 = \ln(6x-8) \quad\Rightarrow\quad x^2 = 6x - 8 \quad\Rightarrow$
$x^2 - 6x + 8 = 0 \quad\Rightarrow\quad (x-2)(x-4) = 0 \quad\Rightarrow\quad x = 2$ or $x = 4$, both are valid solutions.

36. $\ln(2x+1) = 2 - \ln x \quad\Rightarrow\quad \ln x + \ln(2x+1) = \ln e^2 \quad\Rightarrow\quad \ln[x(2x+1)] = \ln e^2 \quad\Rightarrow\quad 2x^2 + x = e^2 \quad\Rightarrow$
$2x^2 + x - e^2 = 0 \quad\Rightarrow\quad x = \dfrac{-1 + \sqrt{1 + 8e^2}}{4}$ [since $x > 0$].

37. $e^{ax} = Ce^{bx} \quad\Leftrightarrow\quad \ln e^{ax} = \ln[C(e^{bx})] \quad\Leftrightarrow\quad ax = \ln C + bx + \ln e^{bx} \quad\Leftrightarrow\quad ax = \ln C + bx \quad\Leftrightarrow$
$ax - bx = \ln C \quad\Leftrightarrow\quad (a-b)x = \ln C \quad\Leftrightarrow\quad x = \dfrac{\ln C}{a-b}$

38. $7e^x - e^{2x} = 12 \quad\Leftrightarrow\quad (e^x)^2 - 7e^x + 12 = 0 \quad\Leftrightarrow\quad (e^x - 3)(e^x - 4) = 0$, so we have either $e^x = 3 \quad\Leftrightarrow$
$x = \ln 3$, or $e^x = 4 \quad\Leftrightarrow\quad x = \ln 4$.

39. $e^{2+5x} = 100 \quad\Rightarrow\quad \ln\left(e^{2+5x}\right) = \ln 100 \quad\Rightarrow\quad 2 + 5x = \ln 100 \quad\Rightarrow\quad 5x = \ln 100 - 2 \quad\Rightarrow$
$x = \frac{1}{5}(\ln 100 - 2) \approx 0.5210$

40. $\ln(1 + \sqrt{x}) = 2 \quad\Rightarrow\quad 1 + \sqrt{x} = e^2 \quad\Rightarrow\quad \sqrt{x} = e^2 - 1 \quad\Rightarrow\quad x = (e^2 - 1)^2 \approx 40.8200$

41. $\ln(e^x - 2) = 3 \quad\Rightarrow\quad e^x - 2 = e^3 \quad\Rightarrow\quad e^x = e^3 + 2 \quad\Rightarrow\quad x = \ln(e^3 + 2) \approx 3.0949$

42. $3^{1/(x-4)} = 7 \quad\Rightarrow\quad \ln 3^{1/(x-4)} = \ln 7 \quad\Rightarrow\quad \dfrac{1}{x-4}\ln 3 = \ln 7 \quad\Rightarrow\quad x - 4 = \dfrac{\ln 3}{\ln 7} \quad\Rightarrow$
$x = 4 + \dfrac{\ln 3}{\ln 7} \approx 4.5646$

43. (a) $e^x < 10 \quad\Rightarrow\quad \ln e^x < \ln 10 \quad\Rightarrow\quad x < \ln 10 \quad\Rightarrow\quad x \in (-\infty, \ln 10)$

(b) $\ln x > -1 \quad\Rightarrow\quad e^{\ln x} > e^{-1} \quad\Rightarrow\quad x > e^{-1} \quad\Rightarrow\quad x \in (1/e, \infty)$

44. (a) $2 < \ln x < 9 \quad\Rightarrow\quad e^2 < e^{\ln x} < e^9 \quad\Rightarrow\quad e^2 < x < e^9 \quad\Rightarrow\quad x \in (e^2, e^9)$

(b) $e^{2-3x} > 4 \quad\Rightarrow\quad \ln e^{2-3x} > \ln 4 \quad\Rightarrow\quad 2 - 3x > \ln 4 \quad\Rightarrow\quad -3x > \ln 4 - 2 \quad\Rightarrow$
$x < -\frac{1}{3}(\ln 4 - 2) \quad\Rightarrow\quad x \in \left(-\infty, \frac{1}{3}(2 - \ln 4)\right)$

45. 3 ft = 36 in, so we need x such that $\log_2 x = 36 \quad\Leftrightarrow\quad x = 2^{36} = 68{,}719{,}476{,}736$. In miles, this is
$68{,}719{,}476{,}736 \text{ in} \cdot \dfrac{1 \text{ ft}}{12 \text{ in}} \cdot \dfrac{1 \text{ mi}}{5280 \text{ ft}} \approx 1{,}084{,}587.7$ mi.