7.5 Inverse Trigonometric Functions

1. (a) $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$ since $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$ and $\frac{\pi}{3}$ is in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

(b) $\cos^{-1}(-1) = \pi$ since $\cos\pi = -1$ and π is in $[0, \pi]$.

2. (a) $\arctan(-1) = -\frac{\pi}{4}$ since $\tan\left(-\frac{\pi}{4}\right) = -1$ and $-\frac{\pi}{4}$ is in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

(b) $\csc^{-1} 2 = \frac{\pi}{6}$ since $\csc\frac{\pi}{6} = 2$ and $\frac{\pi}{6}$ is in $\left(0, \frac{\pi}{2}\right] \cup \left(\pi, \frac{3\pi}{2}\right]$.

3. (a) $\tan^{-1}\sqrt{3} = \frac{\pi}{3}$ since $\tan\frac{\pi}{3} = \sqrt{3}$ and $\frac{\pi}{3}$ is in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

(b) $\arcsin\left(-\frac{1}{\sqrt{2}}\right) = -\frac{\pi}{4}$ since $\sin\left(-\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$ and $-\frac{\pi}{4}$ is in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

4. (a) $\sec^{-1}\sqrt{2} = \frac{\pi}{4}$ since $\sec\frac{\pi}{4} = \sqrt{2}$ and $\frac{\pi}{4}$ is in $\left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$.

(b) $\arcsin 1 = \frac{\pi}{2}$ since $\sin\frac{\pi}{2} = 1$ and $\frac{\pi}{2}$ is in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

5. (a) $\arccos(\cos 2\pi) = \arccos(1) = 0$

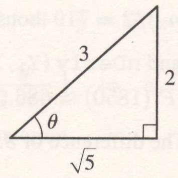
(b) $\tan(\tan^{-1} 5) = 5$

6. (a) $\tan^{-1}\left(\tan\dfrac{3\pi}{4}\right) = \tan^{-1}(-1) = -\dfrac{\pi}{4}$

(b) $\cos\left(\arcsin\dfrac{1}{2}\right) = \cos\left(\dfrac{\pi}{6}\right) = \dfrac{\sqrt{3}}{2}$

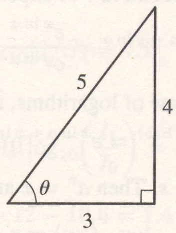
7. Let $\theta = \sin^{-1}\left(\frac{2}{3}\right)$.

Then $\tan\left(\sin^{-1}\left(\frac{2}{3}\right)\right) = \tan\theta = \dfrac{2}{\sqrt{5}}$.



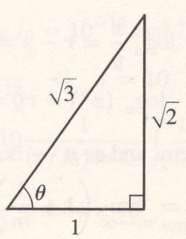
8. Let $\theta = \arccos\frac{3}{5}$.

Then $\csc\left(\arccos\left(\frac{3}{5}\right)\right) = \csc\theta = \frac{5}{4}$.



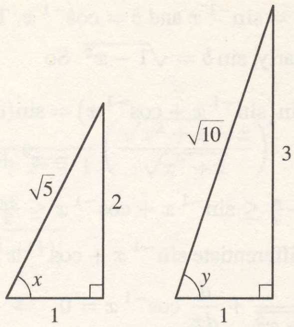
9. Let $\theta = \tan^{-1}\sqrt{2}$. Then

$$\sin\left(2\tan^{-1}\sqrt{2}\right) = \sin(2\theta) = 2\sin\theta\cos\theta$$
$$= 2\left(\frac{\sqrt{2}}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{3}}\right) = \frac{2\sqrt{2}}{3}$$



10. Let $x = \tan^{-1} 2$ and $y = \tan^{-1} 3$. Then

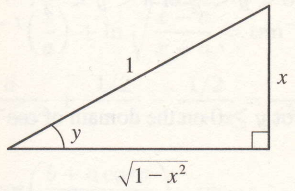
$$\cos\left(\tan^{-1} 2 + \tan^{-1} 3\right) = \cos(x + y) = \cos x \cos y - \sin x \sin y$$
$$= \frac{1}{\sqrt{5}}\frac{1}{\sqrt{10}} - \frac{2}{\sqrt{5}}\frac{3}{\sqrt{10}}$$
$$= \frac{-5}{\sqrt{50}} = \frac{-5}{5\sqrt{2}} = \frac{-1}{\sqrt{2}}$$



11. Let $y = \sin^{-1} x$. Then $-\frac{\pi}{2} \le y \le \frac{\pi}{2} \Rightarrow \cos y \ge 0$, so $\cos(\sin^{-1} x) = \cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2}$

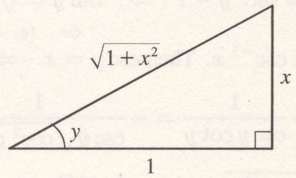
12. Let $y = \sin^{-1} x$. Then $\sin y = x$, so from the triangle we see that

$$\tan(\sin^{-1} x) = \tan y = \frac{x}{\sqrt{1 - x^2}}.$$



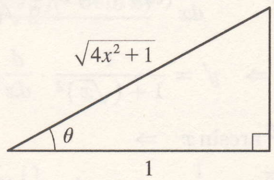
13. Let $y = \tan^{-1} x$. Then $\tan y = x$, so from the triangle we see that

$$\sin(\tan^{-1} x) = \sin y = \frac{x}{\sqrt{1 + x^2}}.$$

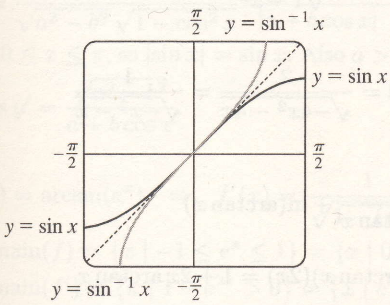


14. Let $\theta = \arctan 2x$. Then $\tan\theta = 2x$, so from the diagram we see that

$$\csc(\arctan 2x) = \csc\theta = \frac{\sqrt{4x^2 + 1}}{2x}.$$

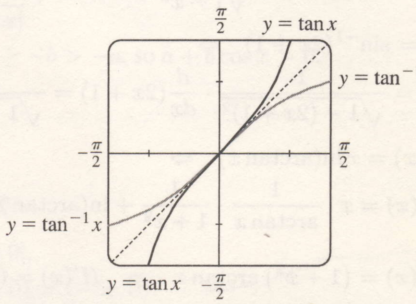


15.



The graph of $\sin^{-1} x$ is the reflection of the graph of $\sin x$ about the line $y = x$.

16.



The graph of $\tan^{-1} x$ is the reflection of the graph of $\tan x$ about the line $y = x$.

17. Let $y = \cos^{-1} x$. Then $\cos y = x$ and $0 \le y \le \pi \Rightarrow -\sin y \dfrac{dy}{dx} = 1 \Rightarrow$

$$\frac{dy}{dx} = -\frac{1}{\sin y} = -\frac{1}{\sqrt{1 - \cos^2 y}} = -\frac{1}{\sqrt{1 - x^2}}.$$ [Note that $\sin y \ge 0$ for $0 \le y \le \pi$.]

18. (a) Let $a = \sin^{-1} x$ and $b = \cos^{-1} x$. Then $\cos a = \sqrt{1 - \sin^2 a} = \sqrt{1 - x^2}$ since $\cos a \geq 0$ for $-\frac{\pi}{2} \leq a \leq \frac{\pi}{2}$.

Similarly, $\sin b = \sqrt{1 - x^2}$. So

$$\begin{aligned}\sin\left(\sin^{-1} x + \cos^{-1} x\right) &= \sin(a + b) = \sin a \cos b + \cos a \sin b = x \cdot x + \sqrt{1 - x^2}\,\sqrt{1 - x^2} \\ &= x^2 + (1 - x^2) = 1\end{aligned}$$

But $-\frac{\pi}{2} \leq \sin^{-1} x + \cos^{-1} x \leq \frac{3\pi}{2}$, and so $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$.

(b) We differentiate $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$ with respect to x, and get

$$\frac{1}{\sqrt{1 - x^2}} + \frac{d}{dx} \cos^{-1} x = 0 \quad \Rightarrow \quad \frac{d}{dx} \cos^{-1} x = -\frac{1}{\sqrt{1 - x^2}}.$$

19. Let $y = \cot^{-1} x$. Then $\cot y = x \;\Rightarrow\; -\csc^2 y \,\dfrac{dy}{dx} = 1 \;\Rightarrow\; \dfrac{dy}{dx} = -\dfrac{1}{\csc^2 y} = -\dfrac{1}{1 + \cot^2 y} = -\dfrac{1}{1 + x^2}$.

20. Let $y = \sec^{-1} x$. Then $\sec y = x$ and $y \in \left(0, \frac{\pi}{2}\right] \cup \left[\pi, \frac{3\pi}{2}\right)$. Differentiate with respect to x:

$\sec y \,\tan y \left(\dfrac{dy}{dx}\right) = 1 \;\Rightarrow\; \dfrac{dy}{dx} = \dfrac{1}{\sec y\,\tan y} = \dfrac{1}{\sec y\,\sqrt{\sec^2 y - 1}} = \dfrac{1}{x\,\sqrt{x^2 - 1}}$. Note that

$\tan^2 y = \sec^2 y - 1 \;\Rightarrow\; \tan y = \sqrt{\sec^2 y - 1}$ since $\tan y > 0$ when $0 < y < \frac{\pi}{2}$ or $\pi < y < \frac{3\pi}{2}$.

21. Let $y = \csc^{-1} x$. Then $\csc y = x \;\Rightarrow\; -\csc y \cot y \,\dfrac{dy}{dx} = 1 \;\Rightarrow$

$\dfrac{dy}{dx} = -\dfrac{1}{\csc y \cot y} = -\dfrac{1}{\csc y \sqrt{\csc^2 y - 1}} = -\dfrac{1}{x\sqrt{x^2 - 1}}$. Note that $\cot y \geq 0$ on the domain of $\csc^{-1} x$.

22. $y = \sqrt{\tan^{-1} x} = (\tan^{-1} x)^{1/2} \;\Rightarrow$

$y' = \frac{1}{2}(\tan^{-1} x)^{-1/2} \cdot \dfrac{d}{dx}(\tan^{-1} x) = \dfrac{1}{2\sqrt{\tan^{-1} x}} \cdot \dfrac{1}{1 + x^2} = \dfrac{1}{2\sqrt{\tan^{-1} x}\,(1 + x^2)}$

23. $y = \tan^{-1} \sqrt{x} \;\Rightarrow\; y' = \dfrac{1}{1 + (\sqrt{x}\,)^2} \cdot \dfrac{d}{dx}(\sqrt{x}\,) = \dfrac{1}{1 + x}\left(\frac{1}{2} x^{-1/2}\right) = \dfrac{1}{2\sqrt{x}\,(1 + x)}$

24. $h(x) = \sqrt{1 - x^2} \arcsin x \;\Rightarrow$

$h'(x) = \sqrt{1 - x^2} \cdot \dfrac{1}{\sqrt{1 - x^2}} + \arcsin x \left[\frac{1}{2}(1 - x^2)^{-1/2}(-2x)\right] = 1 - \dfrac{x \arcsin x}{\sqrt{1 - x^2}}$

25. $y = \sin^{-1}(2x + 1) \;\Rightarrow$

$y' = \dfrac{1}{\sqrt{1 - (2x + 1)^2}} \cdot \dfrac{d}{dx}(2x + 1) = \dfrac{1}{\sqrt{1 - (4x^2 + 4x + 1)}} \cdot 2 = \dfrac{2}{\sqrt{-4x^2 - 4x}} = \dfrac{1}{\sqrt{-x^2 - x}}$

26. $f(x) = x \ln(\arctan x) \;\Rightarrow$

$f'(x) = x \cdot \dfrac{1}{\arctan x} \cdot \dfrac{1}{1 + x^2} + \ln(\arctan x) \cdot 1 = \dfrac{x}{(1 + x^2)\arctan x} + \ln(\arctan x)$

27. $H(x) = (1 + x^2)\arctan x \;\Rightarrow\; H'(x) = (1 + x^2)\dfrac{1}{1 + x^2} + (\arctan x)(2x) = 1 + 2x \arctan x$

28. $h(t) = e^{\sec^{-1} t} \;\Rightarrow\; h'(t) = e^{\sec^{-1} t} \dfrac{d}{dt}\left(\sec^{-1} t\right) = \dfrac{e^{\sec^{-1} t}}{t\sqrt{t^2 - 1}}$

29. $y = \cos^{-1}(e^{2x}) \;\Rightarrow\; y' = -\dfrac{1}{\sqrt{1 - (e^{2x})^2}} \cdot \dfrac{d}{dx}\left(e^{2x}\right) = -\dfrac{2e^{2x}}{\sqrt{1 - e^{4x}}}$

30. $y = x \cos^{-1} x - \sqrt{1 - x^2} \;\Rightarrow\; y' = \cos^{-1} x - \dfrac{x}{\sqrt{1 - x^2}} + \dfrac{x}{\sqrt{1 - x^2}} = \cos^{-1} x$

31. $y = \arctan(\cos\theta) \quad\Rightarrow\quad y' = \dfrac{1}{1+(\cos\theta)^2}(-\sin\theta) = -\dfrac{\sin\theta}{1+\cos^2\theta}$

32. $y = \tan^{-1}\left(x - \sqrt{x^2+1}\right) \quad\Rightarrow$

$$y' = \frac{1}{1+\left(x-\sqrt{x^2+1}\right)^2}\left(1 - \frac{x}{\sqrt{x^2+1}}\right) = \frac{1}{1+x^2-2x\sqrt{x^2+1}+x^2+1}\left(\frac{\sqrt{x^2+1}-x}{\sqrt{x^2+1}}\right)$$

$$= \frac{\sqrt{x^2+1}-x}{2\left(1+x^2-x\sqrt{x^2+1}\right)\sqrt{x^2+1}} = \frac{\sqrt{x^2+1}-x}{2\left[\sqrt{x^2+1}\,(1+x^2)-x(x^2+1)\right]}$$

$$= \frac{\sqrt{x^2+1}-x}{2\left[(1+x^2)\left(\sqrt{x^2+1}-x\right)\right]} = \frac{1}{2(1+x^2)}$$

33. $h(t) = \cot^{-1}(t) + \cot^{-1}(1/t) \quad\Rightarrow$

$$h'(t) = -\frac{1}{1+t^2} - \frac{1}{1+(1/t)^2}\cdot\frac{d}{dt}\frac{1}{t} = -\frac{1}{1+t^2} - \frac{t^2}{t^2+1}\left(-\frac{1}{t^2}\right) = -\frac{1}{1+t^2} + \frac{1}{t^2+1} = 0.$$

Note that this makes sense because $h(t) = \dfrac{\pi}{2}$ for $t > 0$ and $h(t) = -\dfrac{\pi}{2}$ for $t < 0$.

34. $y = \tan^{-1}\left(\dfrac{x}{a}\right) + \ln\sqrt{\dfrac{x-a}{x+a}} = \tan^{-1}\left(\dfrac{x}{a}\right) + \frac{1}{2}\ln(x-a) - \frac{1}{2}\ln(x+a) \quad\Rightarrow$

$$y' = \frac{a}{x^2+a^2} + \frac{1/2}{x-a} - \frac{1/2}{x+a} = \frac{a}{x^2+a^2} + \frac{a}{x^2-a^2} = \frac{2ax^2}{x^4-a^4}$$

35. $y = \arccos\left(\dfrac{b+a\cos x}{a+b\cos x}\right) \quad\Rightarrow$

$$y' = -\frac{1}{\sqrt{1-\left(\dfrac{b+a\cos x}{a+b\cos x}\right)^2}}\;\frac{(a+b\cos x)(-a\sin x)-(b+a\cos x)(-b\sin x)}{(a+b\cos x)^2}$$

$$= \frac{1}{\sqrt{a^2+b^2\cos^2 x-b^2-a^2\cos^2 x}}\;\frac{(a^2-b^2)\sin x}{|a+b\cos x|}$$

$$= \frac{1}{\sqrt{a^2-b^2}\,\sqrt{1-\cos^2 x}}\;\frac{(a^2-b^2)\sin x}{|a+b\cos x|} = \frac{\sqrt{a^2-b^2}}{|a+b\cos x|}\;\frac{\sin x}{|\sin x|}$$

But $0 \le x \le \pi$, so $|\sin x| = \sin x$. Also $a > b > 0 \;\Rightarrow\; b\cos x \ge -b > -a$, so $a + b\cos x > 0$.

Thus $y' = \dfrac{\sqrt{a^2-b^2}}{a+b\cos x}$.

36. $f(x) = \arcsin(e^x) \quad\Rightarrow\quad f'(x) = \dfrac{1}{\sqrt{1-(e^x)^2}}\cdot e^x = \dfrac{e^x}{\sqrt{1-e^{2x}}}$.

$\text{Domain}(f) = \{x \mid -1 \le e^x \le 1\} = \{x \mid 0 < e^x \le 1\} = (-\infty, 0]$.

$\text{Domain}(f') = \{x \mid 1 - e^{2x} > 0\} = \{x \mid e^{2x} < 1\} = \{x \mid 2x < 0\} = (-\infty, 0)$.

37. $g(x) = \cos^{-1}(3-2x) \quad\Rightarrow\quad g'(x) = -\dfrac{1}{\sqrt{1-(3-2x)^2}}(-2) = \dfrac{2}{\sqrt{1-(3-2x)^2}}$.

$\text{Domain}(g) = \{x \mid -1 \le 3-2x \le 1\} = \{x \mid -4 \le -2x \le -2\} = \{x \mid 2 \ge x \ge 1\} = [1, 2]$.

$\text{Domain}(g') = \{x \mid 1-(3-2x)^2 > 0\} = \{x \mid (3-2x)^2 < 1\} = \{x \mid |3-2x| < 1\}$

$= \{x \mid -1 < 3-2x < 1\} = \{x \mid -4 < -2x < -2\} = \{x \mid 2 > x > 1\} = (1, 2)$

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