7 ☐ INVERSE FUNCTIONS:
Exponential, Logarithmic, and Inverse Trigonometric Functions

7.1 Inverse Functions

1. (a) See Definition 1.

(b) It must pass the Horizontal Line Test.

2. (a) $f^{-1}(y) = x \quad \Leftrightarrow \quad f(x) = y$ for any y in B. The domain of f^{-1} is B and the range of f^{-1} is A.

(b) See the steps in (5).

(c) Reflect the graph of f about the line $y = x$.

3. f is not one-to-one because $2 \neq 6$, but $f(2) = 2.0 = f(6)$.

4. f is one-to-one since for any two different domain values, there are different range values.

5. No horizontal line intersects the graph of f more than once. Thus, by the Horizontal Line Test, f is one-to-one.

6. The horizontal line $y = 0$ (the x-axis) intersects the graph of f in more than one point. Thus, by the Horizontal Line Test, f is not one-to-one.

7. The horizontal line $y = 0$ (the x-axis) intersects the graph of f in more than one point. Thus, by the Horizontal Line Test, f is not one-to-one.

8. No horizontal line intersects the graph of f more than once. Thus, by the Horizontal Line Test, f is one-to-one.

9. The graph of $f(x) = \frac{1}{2}(x + 5)$ is a line with slope $\frac{1}{2}$. It passes the Horizontal Line Test, so f is one-to-one.
Algebraic solution: If $x_1 \neq x_2$, then $x_1 + 5 \neq x_2 + 5 \quad \Rightarrow \quad \frac{1}{2}(x_1 + 5) \neq \frac{1}{2}(x_2 + 5) \quad \Rightarrow \quad f(x_1) \neq f(x_2)$, so f is one-to-one.

10. The graph of $f(x) = 1 + 4x - x^2$ is a parabola with axis of symmetry $x = -\frac{b}{2a} = -\frac{4}{2(-1)} = 2$. Pick any x-values equidistant from 2 to find two equal function values. For example, $f(1) = 4$ and $f(3) = 4$, so f is not 1-1.

11. $x_1 \neq x_2 \quad \Rightarrow \quad \sqrt{x_1} \neq \sqrt{x_2} \quad \Rightarrow \quad g(x_1) \neq g(x_2)$, so g is 1-1.

12. $g(x) = |x| \quad \Rightarrow \quad g(-1) = 1 = g(1)$, so g is not one-to-one.

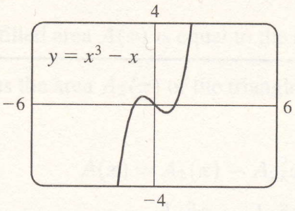
13. $h(x) = x^4 + 5 \quad \Rightarrow \quad h(1) = 6 = h(-1)$, so h is not 1-1.

14. $x_1 \neq x_2 \quad \Rightarrow \quad x_1^4 \neq x_2^4 \quad [\text{since } x \geq 0] \quad \Rightarrow \quad x_1^4 + 5 \neq x_2^4 + 5 \quad \Rightarrow \quad h(x_1) \neq h(x_2)$, so h is 1-1.

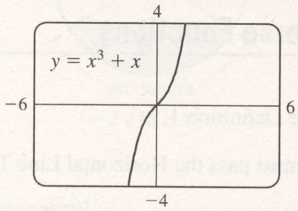
15. A football will attain every height h up to its maximum height twice: once on the way up, and again on the way down. Thus, even if t_1 does not equal t_2, $f(t_1)$ may equal $f(t_2)$, so f is not 1-1.

16. f is not 1-1 because eventually we all stop growing and therefore, there are two times at which we have the same height.

17. f does not pass the Horizontal Line Test, so f is not 1-1.



18. f passes the Horizontal Line Test, so f is 1-1.



19. Since $f(2) = 9$ and f is 1-1, we know that $f^{-1}(9) = 2$. Remember, if the point $(2, 9)$ is on the graph of f, then the point $(9, 2)$ is on the graph of f^{-1}.

20. $f(x) = x + \cos x \;\Rightarrow\; f'(x) = 1 - \sin x \geq 0$, with equality only if $x = \frac{\pi}{2} + 2n\pi$. So f is increasing on $\mathbb{R}$, and hence, 1-1. By inspection, $f(0) = 0 + \cos 0 = 1$, so $f^{-1}(1) = 0$.

21. $h(x) = x + \sqrt{x} \;\Rightarrow\; h'(x) = 1 + 1/(2\sqrt{x}) > 0$ on $(0, \infty)$. So h is increasing and hence, 1-1. By inspection, $h(4) = 4 + \sqrt{4} = 6$, so $h^{-1}(6) = 4$.

22. (a) f is 1-1 because it passes the Horizontal Line Test.

(b) Domain of $f = [-3, 3] =$ Range of f^{-1}. Range of $f = [-2, 2] =$ Domain of f^{-1}.

(c) Since $f(-2) = 1$, $f^{-1}(1) = -2$.

23. We solve $C = \frac{5}{9}(F - 32)$ for F: $\frac{9}{5}C = F - 32 \;\Rightarrow\; F = \frac{9}{5}C + 32$. This gives us a formula for the inverse function, that is, the Fahrenheit temperature F as a function of the Celsius temperature C. $F \geq -459.67 \;\Rightarrow$ $\frac{9}{5}C + 32 \geq -459.67 \;\Rightarrow\; \frac{9}{5}C \geq -491.67 \;\Rightarrow\; C \geq -273.15$, the domain of the inverse function.

24. $m = \dfrac{m_0}{\sqrt{1 - v^2/c^2}} \;\Rightarrow\; 1 - \dfrac{v^2}{c^2} = \dfrac{m_0^2}{m^2} \;\Rightarrow\; \dfrac{v^2}{c^2} = 1 - \dfrac{m_0^2}{m^2} \;\Rightarrow\; v^2 = c^2\left(1 - \dfrac{m_0^2}{m^2}\right) \;\Rightarrow$ $v = c\sqrt{1 - \dfrac{m_0^2}{m^2}}$. This formula gives us the speed v of the particle in terms of its mass m, that is, $v = f^{-1}(m)$.

25. $y = f(x) = 3 - 2x \;\Rightarrow\; 2x = 3 - y \;\Rightarrow\; x = \dfrac{3 - y}{2}$. Interchange x and y: $y = \dfrac{3 - x}{2}$. So $f^{-1}(x) = \dfrac{3 - x}{2}$.

26. $f(x) = \dfrac{4x - 1}{2x + 3} \;\Rightarrow\; y = \dfrac{4x - 1}{2x + 3} \;\Rightarrow\; y(2x + 3) = 4x - 1 \;\Rightarrow\; 2xy + 3y = 4x - 1 \;\Rightarrow$ $3y + 1 = 4x - 2xy \;\Rightarrow\; 3y + 1 = (4 - 2y)x \;\Rightarrow\; x = \dfrac{3y + 1}{4 - 2y}$. Interchange x and y: $y = \dfrac{3x + 1}{4 - 2x}$. So $f^{-1}(x) = \dfrac{3x + 1}{4 - 2x}$.

27. $f(x) = \sqrt{10 - 3x} \;\Rightarrow\; y = \sqrt{10 - 3x} \quad (y \geq 0) \;\Rightarrow\; y^2 = 10 - 3x \;\Rightarrow\; 3x = 10 - y^2 \;\Rightarrow$ $x = -\frac{1}{3}y^2 + \frac{10}{3}$. Interchange x and y: $y = -\frac{1}{3}x^2 + \frac{10}{3}$. So $f^{-1}(x) = -\frac{1}{3}x^2 + \frac{10}{3}$. Note that the domain of f^{-1} is $x \geq 0$.

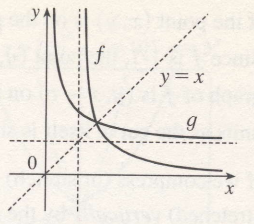
28. $y = f(x) = 2x^3 + 3 \;\Rightarrow\; y - 3 = 2x^3 \;\Rightarrow\; \dfrac{y - 3}{2} = x^3 \;\Rightarrow\; x = \sqrt[3]{\dfrac{y - 3}{2}}$.

Interchange x and y: $y = \sqrt[3]{\dfrac{x - 3}{2}}$. So $f^{-1}(x) = \sqrt[3]{\dfrac{x - 3}{2}}$.

(c) $y = 1/(x-1) \Rightarrow x - 1 = 1/y \Rightarrow x = 1 + 1/y$. Interchange x and y: $y = 1 + 1/x$. So $g(x) = 1 + 1/x$, $x > 0$ (since $y > 1$). Domain $= (0, \infty)$, range $= (1, \infty)$.

(d) $g'(x) = -1/x^2$, so $g'(2) = -\frac{1}{4}$.

(e)



39. $f(0) = 1 \Rightarrow f^{-1}(1) = 0$, and $f(x) = x^3 + x + 1 \Rightarrow f'(x) = 3x^2 + 1$ and $f'(0) = 1$. Thus,
$(f^{-1})'(1) = \dfrac{1}{f'(f^{-1}(1))} = \dfrac{1}{f'(0)} = \dfrac{1}{1} = 1$.

40. $f(1) = 2 \Rightarrow f^{-1}(2) = 1$, and $f(x) = x^5 - x^3 + 2x \Rightarrow f'(x) = 5x^4 - 3x^2 + 2$ and $f'(1) = 4$. Thus,
$(f^{-1})'(2) = \dfrac{1}{f'(f^{-1}(2))} = \dfrac{1}{f'(1)} = \dfrac{1}{4}$.

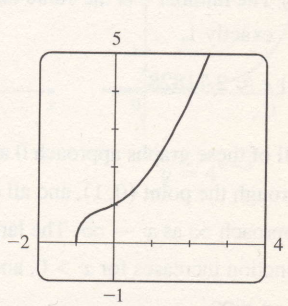
41. $f(0) = 3 \Rightarrow f^{-1}(3) = 0$, and $f(x) = 3 + x^2 + \tan(\pi x/2) \Rightarrow f'(x) = 2x + \frac{\pi}{2}\sec^2(\pi x/2)$ and
$f'(0) = \frac{\pi}{2} \cdot 1 = \frac{\pi}{2}$. Thus, $(f^{-1})'(3) = 1/f'(f^{-1}(3)) = 1/f'(0) = 2/\pi$.

42. $f(1) = 2 \Rightarrow f^{-1}(2) = 1$, and $f(x) = \sqrt{x^3 + x^2 + x + 1} \Rightarrow f'(x) = \dfrac{3x^2 + 2x + 1}{2\sqrt{x^3 + x^2 + x + 1}}$ and
$f'(1) = \dfrac{3 + 2 + 1}{2\sqrt{1 + 1 + 1 + 1}} = \dfrac{3}{2}$. Thus, $(f^{-1})'(2) = 1/f'(f^{-1}(2)) = 1/f'(1) = \frac{2}{3}$.

43. $f(4) = 5 \Rightarrow g(5) = 4$. Thus, $g'(5) = \dfrac{1}{f'(g(5))} = \dfrac{1}{f'(4)} = \dfrac{1}{2/3} = \dfrac{3}{2}$.

44. $f(3) = 2 \Rightarrow g(2) = 3$. Thus, $g'(2) = \dfrac{1}{f'(g(2))} = \dfrac{1}{f'(3)} = 9$. Hence, $G(x) = \dfrac{1}{g(x)} \Rightarrow$
$G'(x) = -\dfrac{g'(x)}{[g(x)]^2} \Rightarrow G'(2) = -\dfrac{g'(2)}{[g(2)]^2} = -\dfrac{9}{(3)^2} = -1$.

45. We see that the graph of $y = f(x) = \sqrt{x^3 + x^2 + x + 1}$ is increasing, so f is 1-1. Enter $x = \sqrt{y^3 + y^2 + y + 1}$ and use your CAS to solve the equation for y. Using Derive, we get two (irrelevant) solutions involving imaginary expressions, as well as one which can be simplified to the following:



$$y = f^{-1}(x) = -\frac{\sqrt[3]{4}}{6}\left(\sqrt[3]{D - 27x^2 + 20} - \sqrt[3]{D + 27x^2 - 20} + \sqrt[3]{2}\right)$$

where $D = 3\sqrt{3}\sqrt{27x^4 - 40x^2 + 16}$. Maple and Mathematica each give two complex expressions and one real expression, and the real expression is equivalent to that given by Derive. For example, Maple's expression simplifies to $\dfrac{1}{6}\dfrac{M^{2/3} - 8 - 2M^{1/3}}{2M^{1/3}}$, where $M = 108x^2 + 12\sqrt{48 - 120x^2 + 81x^4} - 80$.

46. Since $\sin(2n\pi) = 0$, $h(x) = \sin x$ is not one-to-one. $h'(x) = \cos x > 0$ on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, so h is increasing and hence 1-1 on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Let $y = f^{-1}(x) = \sin^{-1} x$ so that $\sin y = x$. Differentiating $\sin y = x$ implicitly with respect to x gives us $\cos y \dfrac{dy}{dx} = 1 \Rightarrow \dfrac{dy}{dx} = \dfrac{1}{\cos y}$. Now $\cos^2 y + \sin^2 y = 1 \Rightarrow \cos y = \pm\sqrt{1 - \sin^2 y}$, but since $\cos y > 0$ on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, we have $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{1 - \sin^2 y}} = \dfrac{1}{\sqrt{1 - x^2}}$.