We use a procedure like long division:

$$\begin{array}{r}
x + \tfrac{1}{3}x^3 + \tfrac{2}{15}x^5 + \cdots \\
1 - \tfrac{1}{2}x^2 + \tfrac{1}{24}x^4 - \cdots \overline{\big)\, x - \tfrac{1}{6}x^3 + \tfrac{1}{120}x^5 - \cdots} \\
x - \tfrac{1}{2}x^3 + \tfrac{1}{24}x^5 - \cdots \\
\hline
\tfrac{1}{3}x^3 - \tfrac{1}{30}x^5 + \cdots \\
\tfrac{1}{3}x^3 - \tfrac{1}{6}x^5 + \cdots \\
\hline
\tfrac{2}{15}x^5 + \cdots
\end{array}$$

Thus $$\tan x = x + \tfrac{1}{3}x^3 + \tfrac{2}{15}x^5 + \cdots$$

Although we have not attempted to justify the formal manipulations used in Example 10, they are legitimate. There is a theorem which states that if both $f(x) = \Sigma\, c_n x^n$ and $g(x) = \Sigma\, b_n x^n$ converge for $|x| < R$ and the series are multiplied as if they were polynomials, then the resulting series also converges for $|x| < R$ and represents $f(x)g(x)$. For division we require $b_0 \neq 0$; the resulting series converges for sufficiently small $|x|$.

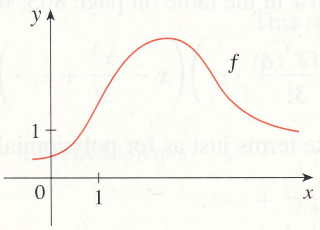
12.10 Exercises

1. If $f(x) = \Sigma_{n=0}^{\infty} b_n(x-5)^n$ for all x, write a formula for b_8.

2. (a) The graph of f is shown. Explain why the series

$$1.6 - 0.8(x-1) + 0.4(x-1)^2 - 0.1(x-1)^3 + \cdots$$

is *not* the Taylor series of f centered at 1.



(b) Explain why the series

$$2.8 + 0.5(x-2) + 1.5(x-2)^2 - 0.1(x-2)^3 + \cdots$$

is *not* the Taylor series of f centered at 2.

3–10 Find the Maclaurin series for $f(x)$ using the definition of a Maclaurin series. [Assume that f has a power series expansion. Do not show that $R_n(x) \to 0$.] Also find the associated radius of convergence.

3. $f(x) = \cos x$
4. $f(x) = \sin 2x$
5. $f(x) = (1 + x)^{-3}$
6. $f(x) = \ln(1 + x)$
7. $f(x) = e^{5x}$
8. $f(x) = xe^x$
9. $f(x) = \sinh x$
10. $f(x) = \cosh x$

11–18 Find the Taylor series for $f(x)$ centered at the given value of a. [Assume that f has a power series expansion. Do not show that $R_n(x) \to 0$.]

11. $f(x) = 1 + x + x^2, \quad a = 2$
12. $f(x) = x^3, \quad a = -1$
13. $f(x) = e^x, \quad a = 3$
14. $f(x) = \ln x, \quad a = 2$
15. $f(x) = \cos x, \quad a = \pi$
16. $f(x) = \sin x, \quad a = \pi/2$
17. $f(x) = 1/\sqrt{x}, \quad a = 9$
18. $f(x) = x^{-2}, \quad a = 1$

19. Prove that the series obtained in Exercise 3 represents $\cos x$ for all x.
20. Prove that the series obtained in Exercise 16 represents $\sin x$ for all x.
21. Prove that the series obtained in Exercise 9 represents $\sinh x$ for all x.
22. Prove that the series obtained in Exercise 10 represents $\cosh x$ for all x.

23–32 Use a Maclaurin series derived in this section to obtain the Maclaurin series for the given function.

23. $f(x) = \cos \pi x$
24. $f(x) = e^{-x/2}$
25. $f(x) = x \tan^{-1} x$
26. $f(x) = \sin(x^4)$
27. $f(x) = x^2 e^{-x}$
28. $f(x) = x \cos 2x$