(i) If $\int_1^\infty f(x)\,dx$ is convergent, then (4) gives

$$\sum_{i=2}^{n} a_i \leqslant \int_1^n f(x)\,dx \leqslant \int_1^\infty f(x)\,dx$$

since $f(x) \geqslant 0$. Therefore

$$s_n = a_1 + \sum_{i=2}^{n} a_i \leqslant a_1 + \int_1^\infty f(x)\,dx = M, \text{ say}$$

Since $s_n \leqslant M$ for all n, the sequence $\{s_n\}$ is bounded above. Also

$$s_{n+1} = s_n + a_{n+1} \geqslant s_n$$

since $a_{n+1} = f(n+1) \geqslant 0$. Thus, $\{s_n\}$ is an increasing bounded sequence and so it is convergent by the Monotonic Sequence Theorem (12.1.11). This means that $\Sigma\, a_n$ is convergent.

(ii) If $\int_1^\infty f(x)\,dx$ is divergent, then $\int_1^n f(x)\,dx \to \infty$ as $n \to \infty$ because $f(x) \geqslant 0$. But (5) gives

$$\int_1^n f(x)\,dx \leqslant \sum_{i=1}^{n-1} a_i = s_{n-1}$$

and so $s_{n-1} \to \infty$. This implies that $s_n \to \infty$ and so $\Sigma\, a_n$ diverges.

12.3 Exercises

1. Draw a picture to show that

$$\sum_{n=2}^{\infty} \frac{1}{n^{1.3}} < \int_1^\infty \frac{1}{x^{1.3}}\,dx$$

What can you conclude about the series?

2. Suppose f is a continuous positive decreasing function for $x \geqslant 1$ and $a_n = f(n)$. By drawing a picture, rank the following three quantities in increasing order:

$$\int_1^6 f(x)\,dx \qquad \sum_{i=1}^{5} a_i \qquad \sum_{i=2}^{6} a_i$$

3–8 ||| Use the Integral Test to determine whether the series is convergent or divergent.

3. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4}$ **4.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[4]{n}}$ **5.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3n+1}$

6. $\displaystyle\sum_{n=1}^{\infty} e^{-n}$ **7.** $\displaystyle\sum_{n=1}^{\infty} ne^{-n}$ **8.** $\displaystyle\sum_{n=1}^{\infty} \frac{n+2}{n+1}$

9–24 ||| Determine whether the series is convergent or divergent.

9. $\displaystyle\sum_{n=1}^{\infty} \frac{2}{n^{0.85}}$ **10.** $\displaystyle\sum_{n=1}^{\infty} (n^{-1.4} + 3n^{-1.2})$

11. $1 + \dfrac{1}{8} + \dfrac{1}{27} + \dfrac{1}{64} + \dfrac{1}{125} + \cdots$

12. $1 + \dfrac{1}{2\sqrt{2}} + \dfrac{1}{3\sqrt{3}} + \dfrac{1}{4\sqrt{4}} + \dfrac{1}{5\sqrt{5}} + \cdots$

13. $\displaystyle\sum_{n=1}^{\infty} \frac{5 - 2\sqrt{n}}{n^3}$ **14.** $\displaystyle\sum_{n=3}^{\infty} \frac{5}{n-2}$

15. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+4}$ **16.** $\displaystyle\sum_{n=1}^{\infty} \frac{3n+2}{n(n+1)}$

17. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ **18.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2-4n+5}$

19. $\displaystyle\sum_{n=1}^{\infty} ne^{-n^2}$ **20.** $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$

21. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ **22.** $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^4+1}$

23. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3+n}$ **24.** $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n \ln n \ln(\ln n)}$

25–28 ||| Find the values of p for which the series is convergent.

25. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ **26.** $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n \ln n\, [\ln(\ln n)]^p}$