Since $\cos^2 x = 1 - \sin^2 x$, we have

$$\int \sin^n x\,dx = -\cos x \sin^{n-1}x + (n-1)\int \sin^{n-2}x\,dx - (n-1)\int \sin^n x\,dx$$

As in Example 4, we solve this equation for the desired integral by taking the last term on the right side to the left side. Thus, we have

$$n\int \sin^n x\,dx = -\cos x \sin^{n-1}x + (n-1)\int \sin^{n-2}x\,dx$$

or

$$\int \sin^n x\,dx = -\frac{1}{n}\cos x \sin^{n-1}x + \frac{n-1}{n}\int \sin^{n-2}x\,dx$$

The reduction formula (7) is useful because by using it repeatedly we could eventually express $\int \sin^n x\,dx$ in terms of $\int \sin x\,dx$ (if n is odd) or $\int (\sin x)^0\,dx = \int dx$ (if n is even).

8.1 Exercises

1–2 Evaluate the integral using integration by parts with the indicated choices of u and dv.

1. $\int x \ln x\,dx;\quad u = \ln x,\ dv = x\,dx$

2. $\int \theta \sec^2\theta\,d\theta;\quad u = \theta,\ dv = \sec^2\theta\,d\theta$

3–32 Evaluate the integral.

3. $\int x \cos 5x\,dx$

4. $\int xe^{-x}\,dx$

5. $\int re^{r/2}\,dr$

6. $\int t \sin 2t\,dt$

7. $\int x^2 \sin \pi x\,dx$

8. $\int x^2 \cos mx\,dx$

9. $\int \ln(2x+1)\,dx$

10. $\int \sin^{-1}x\,dx$

11. $\int \arctan 4t\,dt$

12. $\int p^5 \ln p\,dp$

13. $\int (\ln x)^2\,dx$

14. $\int t^3 e^t\,dt$

15. $\int e^{2\theta}\sin 3\theta\,d\theta$

16. $\int e^{-\theta}\cos 2\theta\,d\theta$

17. $\int y \sinh y\,dy$

18. $\int y \cosh ay\,dy$

19. $\int_0^{\pi} t \sin 3t\,dt$

20. $\int_0^1 (x^2+1)e^{-x}\,dx$

21. $\int_1^2 \frac{\ln x}{x^2}\,dx$

22. $\int_1^4 \sqrt{t}\ln t\,dt$

23. $\int_0^1 \frac{y}{e^{2y}}\,dy$

24. $\int_{\pi/4}^{\pi/2} x \csc^2 x\,dx$

25. $\int_0^{1/2} \cos^{-1}x\,dx$

26. $\int_0^1 x5^x\,dx$

27. $\int \cos x \ln(\sin x)\,dx$

28. $\int_1^{\sqrt{3}} \arctan(1/x)\,dx$

29. $\int \cos(\ln x)\,dx$

30. $\int_0^1 \frac{r^3}{\sqrt{4+r^2}}\,dr$

31. $\int_1^2 x^4(\ln x)^2\,dx$

32. $\int_0^t e^s \sin(t-s)\,ds$

33–36 First make a substitution and then use integration by parts to evaluate the integral.

33. $\int \sin\sqrt{x}\,dx$

34. $\int_1^4 e^{\sqrt{x}}\,dx$

35. $\int_{\sqrt{\pi/2}}^{\sqrt{\pi}} \theta^3\cos(\theta^2)\,d\theta$

36. $\int x^5 e^{x^2}\,dx$

37–40 Evaluate the indefinite integral. Illustrate, and check that your answer is reasonable, by graphing both the function and its antiderivative (take $C = 0$).

37. $\int x \cos \pi x\,dx$

38. $\int x^{3/2}\ln x\,dx$

39. $\int (2x+3)e^x\,dx$

40. $\int x^3 e^{x^2}\,dx$

41. (a) Use the reduction formula in Example 6 to show that

$$\int \sin^2 x\,dx = \frac{x}{2} - \frac{\sin 2x}{4} + C$$

(b) Use part (a) and the reduction formula to evaluate $\int \sin^4 x\,dx$.

42. (a) Prove the reduction formula

$$\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1}x \sin x + \frac{n-1}{n}\int \cos^{n-2}x\,dx$$

(b) Use part (a) to evaluate $\int \cos^2 x\,dx$.
(c) Use parts (a) and (b) to evaluate $\int \cos^4 x\,dx$.

43. (a) Use the reduction formula in Example 6 to show that

$$\int_0^{\pi/2} \sin^n x\,dx = \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2}x\,dx$$

where $n \geqslant 2$ is an integer.
(b) Use part (a) to evaluate $\int_0^{\pi/2} \sin^3 x\,dx$ and $\int_0^{\pi/2} \sin^5 x\,dx$.
(c) Use part (a) to show that, for odd powers of sine,

$$\int_0^{\pi/2} \sin^{2n+1}x\,dx = \frac{2 \cdot 4 \cdot 6 \cdot \cdots \cdot 2n}{3 \cdot 5 \cdot 7 \cdot \cdots \cdot (2n+1)}$$

44. Prove that, for even powers of sine,

$$\int_0^{\pi/2} \sin^{2n}x\,dx = \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{2 \cdot 4 \cdot 6 \cdot \cdots \cdot 2n}\,\frac{\pi}{2}$$

45–48 Use integration by parts to prove the reduction formula.

45. $\int (\ln x)^n\,dx = x(\ln x)^n - n\int (\ln x)^{n-1}\,dx$

46. $\int x^n e^x\,dx = x^n e^x - n\int x^{n-1}e^x\,dx$

47. $\int (x^2+a^2)^n\,dx$

$$= \frac{x(x^2+a^2)^n}{2n+1} + \frac{2na^2}{2n+1}\int (x^2+a^2)^{n-1}\,dx \quad (n \neq -\tfrac{1}{2})$$

48. $\int \sec^n x\,dx = \dfrac{\tan x \sec^{n-2}x}{n-1} + \dfrac{n-2}{n-1}\int \sec^{n-2}x\,dx \quad (n \neq 1)$

49. Use Exercise 45 to find $\int (\ln x)^3\,dx$.

50. Use Exercise 46 to find $\int x^4 e^x\,dx$.

51–52 Find the area of the region bounded by the given curves.

51. $y = xe^{-0.4x}$, $y = 0$, $x = 5$

52. $y = 5\ln x$, $y = x\ln x$

53–54 Use a graph to find approximate x-coordinates of the points of intersection of the given curves. Then find (approximately) the area of the region bounded by the curves.

53. $y = x\sin x$, $y = (x-2)^2$

54. $y = \arctan 3x$, $y = x/2$

55–58 Use the method of cylindrical shells to find the volume generated by rotating the region bounded by the given curves about the specified axis.

55. $y = \cos(\pi x/2)$, $y = 0$, $0 \leqslant x \leqslant 1$; about the y-axis

56. $y = e^x$, $y = e^{-x}$, $x = 1$; about the y-axis

57. $y = e^{-x}$, $y = 0$, $x = -1$, $x = 0$; about $x = 1$

58. $y = e^x$, $x = 0$, $y = \pi$; about the x-axis

59. Find the average value of $f(x) = x^2 \ln x$ on the interval $[1, 3]$.

60. A rocket accelerates by burning its onboard fuel, so its mass decreases with time. Suppose the initial mass of the rocket at liftoff (including its fuel) is m, the fuel is consumed at rate r, and the exhaust gases are ejected with constant velocity v_e (relative to the rocket). A model for the velocity of the rocket at time t is given by the equation

$$v(t) = -gt - v_e \ln\frac{m - rt}{m}$$

where g is the acceleration due to gravity and t is not too large. If $g = 9.8\ \text{m/s}^2$, $m = 30{,}000$ kg, $r = 160$ kg/s, and $v_e = 3000$ m/s, find the height of the rocket one minute after liftoff.

61. A particle that moves along a straight line has velocity $v(t) = t^2 e^{-t}$ meters per second after t seconds. How far will it travel during the first t seconds?

62. If $f(0) = g(0) = 0$ and f'' and g'' are continuous, show that

$$\int_0^a f(x)g''(x)\,dx = f(a)g'(a) - f'(a)g(a) + \int_0^a f''(x)g(x)\,dx$$

63. Suppose that $f(1) = 2$, $f(4) = 7$, $f'(1) = 5$, $f'(4) = 3$, and f'' is continuous. Find the value of $\int_1^4 xf''(x)\,dx$.

64. (a) Use integration by parts to show that

$$\int f(x)\,dx = xf(x) - \int xf'(x)\,dx$$

(b) If f and g are inverse functions and f' is continuous, prove that

$$\int_a^b f(x)\,dx = bf(b) - af(a) - \int_{f(a)}^{f(b)} g(y)\,dy$$

[*Hint:* Use part (a) and make the substitution $y = f(x)$.]