If we put $n = 1/x$ in Formula 8, then $n \to \infty$ as $x \to 0^+$ and so an alternative expression for e is

$$e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n \qquad (9)$$

7.4 Exercises

1. Explain why the natural logarithmic function $y = \ln x$ is used much more frequently in calculus than the other logarithmic functions $y = \log_a x$.

1–24 ||| Differentiate the function.

2. $f(x) = \ln(x^2 + 10)$

3. $f(\theta) = \ln(\cos\theta)$
4. $f(x) = \cos(\ln x)$

5. $f(x) = \log_2(1 - 3x)$
6. $f(x) = \log_{10}\left(\dfrac{x}{x-1}\right)$

7. $f(x) = \sqrt[5]{\ln x}$
8. $f(x) = \ln\sqrt[5]{x}$

9. $f(x) = \sqrt{x}\,\ln x$
10. $f(t) = \dfrac{1 + \ln t}{1 - \ln t}$

11. $F(t) = \ln\dfrac{(2t+1)^3}{(3t-1)^4}$

12. $h(x) = \ln\left(x + \sqrt{x^2 - 1}\right)$

13. $g(x) = \ln\dfrac{a - x}{a + x}$
14. $F(y) = y\ln(1 + e^y)$

15. $f(u) = \dfrac{\ln u}{1 + \ln(2u)}$
16. $y = \ln(x^4\sin^2 x)$

17. $h(t) = t^3 - 3^t$
18. $y = 10^{\tan\theta}$

19. $y = \ln|2 - x - 5x^2|$

20. $G(u) = \ln\sqrt{\dfrac{3u + 2}{3u - 2}}$

21. $y = \ln(e^{-x} + xe^{-x})$
22. $y = [\ln(1 + e^x)]^2$

23. $y = 5^{-1/x}$
24. $y = 2^{3^{x^2}}$

25–28 ||| Find y' and y''.

25. $y = x\ln x$
26. $y = \dfrac{\ln x}{x^2}$

27. $y = \log_{10} x$
28. $y = \ln(\sec x + \tan x)$

29–32 ||| Differentiate f and find the domain of f.

29. $f(x) = \dfrac{x}{1 - \ln(x - 1)}$
30. $f(x) = \dfrac{1}{1 + \ln x}$

31. $f(x) = x^2\ln(1 - x^2)$
32. $f(x) = \ln\ln\ln x$

33. If $f(x) = \dfrac{x}{\ln x}$, find $f'(e)$.

34. If $f(x) = x^2\ln x$, find $f'(1)$.

35–36 ||| Find an equation of the tangent line to the curve at the given point.

35. $y = \ln\ln x$, $(e, 0)$

36. $y = \ln(x^3 - 7)$, $(2, 0)$

37–38 ||| Find $f'(x)$. Check that your answer is reasonable by comparing the graphs of f and f'.

37. $f(x) = \sin x + \ln x$

38. $f(x) = x^{\cos x}$

39–50 ||| Use logarithmic differentiation to find the derivative of the function.

39. $y = (2x + 1)^5(x^4 - 3)^6$
40. $y = \sqrt{x}\,e^{x^2}(x^2 + 1)^{10}$

41. $y = \dfrac{\sin^2 x \tan^4 x}{(x^2+1)^2}$

42. $y = \sqrt[4]{\dfrac{x^2+1}{x^2-1}}$

43. $y = x^x$

44. $y = x^{1/x}$

45. $y = x^{\sin x}$

46. $y = (\sin x)^x$

47. $y = (\ln x)^x$

48. $y = x^{\ln x}$

49. $y = x^{e^x}$

50. $y = (\ln x)^{\cos x}$

51. Find y' if $y = \ln(x^2 + y^2)$.

52. Find y' if $x^y = y^x$.

53. Find a formula for $f^{(n)}(x)$ if $f(x) = \ln(x-1)$.

54. Find $\dfrac{d^9}{dx^9}(x^8 \ln x)$.

55–56 ||| Use a graph to estimate the roots of the equation. Then use these estimates as the initial approximations in Newton's method to find the roots correct to six decimal places.

55. $(x-4)^2 = \ln x$

56. $\ln(4 - x^2) = x$

57. Find the intervals of concavity and the inflection points of the function $f(x) = (\ln x)/\sqrt{x}$.

58. Find the absolute minimum value of the function $f(x) = x \ln x$.

59–62 ||| Discuss the curve under the guidelines of Section 4.5.

59. $y = \ln(\sin x)$

60. $y = \ln(\tan^2 x)$

61. $y = \ln(1 + x^2)$

62. $y = \ln(x^2 - 3x + 2)$

CAS **63.** If $f(x) = \ln(2x + x\sin x)$, use the graphs of f, f', and f'' to estimate the intervals of increase and the inflection points of f on the interval $(0, 15]$.

64. Investigate the family of curves $f(x) = \ln(x^2 + c)$. What happens to the inflection points and asymptotes as c changes? Graph several members of the family to illustrate what you discover.

65–76 ||| Evaluate the integral.

65. $\displaystyle\int_2^4 \frac{3}{x}\,dx$

66. $\displaystyle\int_1^2 \frac{4+u^2}{u^3}\,du$

67. $\displaystyle\int_1^2 \frac{dt}{8-3t}$

68. $\displaystyle\int_4^9 \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2 dx$

69. $\displaystyle\int_1^e \frac{x^2+x+1}{x}\,dx$

70. $\displaystyle\int_e^6 \frac{dx}{x\ln x}$

71. $\displaystyle\int \frac{2-x^2}{6x-x^3}\,dx$

72. $\displaystyle\int \frac{\cos x}{2+\sin x}\,dx$

73. $\displaystyle\int \frac{(\ln x)^2}{x}\,dx$

74. $\displaystyle\int \frac{e^x}{e^x+1}\,dx$

75. $\displaystyle\int_1^2 10^t\,dt$

76. $\displaystyle\int x2^{x^2}\,dx$

77. Show that $\int \cot x\,dx = \ln|\sin x| + C$ by (a) differentiating the right side of the equation and (b) using the method of Example 11.

78. Find, correct to three decimal places, the area of the region above the hyperbola $y = 2/(x-2)$, below the x-axis, and between the lines $x = -4$ and $x = -1$.

79. Find the volume of the solid obtained by rotating the region under the curve

$$y = \frac{1}{\sqrt{x+1}}$$

from 0 to 1 about the x-axis.

80. Find the volume of the solid obtained by rotating the region under the curve

$$y = \frac{1}{x^2+1}$$

from 0 to 3 about the y-axis.

81. The work done by a gas when it expands from volume V_1 to volume V_2 is $W = \int_{V_1}^{V_2} P\,dV$, where $P = P(V)$ is the pressure as a function of the volume V. (See Exercise 27 in Section 6.4.) Boyle's Law states that when a quantity of gas expands at constant temperature, $PV = C$, where C is a constant. If the initial volume is 600 cm^3 and the initial pressure is 150 kPa, find the work done by the gas when it expands at constant temperature to 1000 cm^3.

82. Find f if $f''(x) = x^{-2}$, $x > 0$, $f(1) = 0$, and $f(2) = 0$.

83. If g is the inverse function of $f(x) = 2x + \ln x$, find $g'(2)$.

84. If $f(x) = e^x + \ln x$ and $h(x) = f^{-1}(x)$, find $h'(e)$.

85. For what values of m do the line $y = mx$ and the curve $y = x/(x^2+1)$ enclose a region? Find the area of the region.

86. (a) Find the linear approximation to $f(x) = \ln x$ near 1.
(b) Illustrate part (a) by graphing f and its linearization.
(c) For what values of x is the linear approximation accurate to within 0.1?

87. Use the definition of derivative to prove that

$$\lim_{x\to 0} \frac{\ln(1+x)}{x} = 1$$

88. Show that $\displaystyle\lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n = e^x$ for any $x > 0$.