gives $du = 2\,dx$, so $dx = du/2$. When $x = 0$, $u = 0$; when $x = \frac{1}{4}$, $u = \frac{1}{2}$. So

$$\int_0^{1/4} \frac{1}{\sqrt{1-4x^2}}\,dx = \tfrac{1}{2}\int_0^{1/2} \frac{du}{\sqrt{1-u^2}} = \tfrac{1}{2}\sin^{-1}u\Big]_0^{1/2}$$

$$= \tfrac{1}{2}\left[\sin^{-1}\left(\tfrac{1}{2}\right) - \sin^{-1}0\right] = \frac{1}{2}\cdot\frac{\pi}{6} = \frac{\pi}{12}$$

EXAMPLE 9 Evaluate $\displaystyle\int \frac{1}{x^2+a^2}\,dx$.

SOLUTION To make the given integral more like Equation 13 we write

$$\int \frac{dx}{x^2+a^2} = \int \frac{dx}{a^2\left(\dfrac{x^2}{a^2}+1\right)} = \frac{1}{a^2}\int \frac{dx}{\left(\dfrac{x}{a}\right)^2+1}$$

This suggests that we substitute $u = x/a$. Then $du = dx/a$, $dx = a\,du$, and

$$\int \frac{dx}{x^2+a^2} = \frac{1}{a^2}\int \frac{a\,du}{u^2+1} = \frac{1}{a}\int \frac{du}{u^2+1} = \frac{1}{a}\tan^{-1}u + C$$

Thus, we have the formula

One of the main uses of inverse trigonometric functions is that they often arise when we integrate rational functions.

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$$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

EXAMPLE 10 Find $\displaystyle\int \frac{x}{x^4+9}\,dx$.

SOLUTION We substitute $u = x^2$ because then $du = 2x\,dx$ and we can use Equation 14 with $a = 3$:

$$\int \frac{x}{x^4+9}\,dx = \frac{1}{2}\int \frac{du}{u^2+9} = \frac{1}{2}\cdot\frac{1}{3}\tan^{-1}\left(\frac{u}{3}\right) + C$$

$$= \frac{1}{6}\tan^{-1}\left(\frac{x^2}{3}\right) + C$$

7.5 Exercises

1–10 ||| Find the exact value of each expression.

1. (a) $\sin^{-1}(\sqrt{3}/2)$ (b) $\cos^{-1}(-1)$
2. (a) $\arctan(-1)$ (b) $\csc^{-1} 2$
3. (a) $\tan^{-1}\sqrt{3}$ (b) $\arcsin(-1/\sqrt{2})$
4. (a) $\sec^{-1}\sqrt{2}$ (b) $\arcsin 1$
5. (a) $\arccos(\cos 2\pi)$ (b) $\tan(\tan^{-1} 5)$
6. (a) $\tan^{-1}(\tan 3\pi/4)$ (b) $\cos(\arcsin \frac{1}{2})$

7. $\tan(\sin^{-1}(\frac{2}{3}))$

8. $\csc(\arccos \frac{3}{5})$

9. $\sin(2\tan^{-1}\sqrt{2})$

10. $\cos(\tan^{-1} 2 + \tan^{-1} 3)$

11. Prove that $\cos(\sin^{-1}x) = \sqrt{1-x^2}$ for $-1 \leqslant x \leqslant 1$.

12–14 ||| Simplify the expression.

12. $\tan(\sin^{-1}x)$

13. $\sin(\tan^{-1}x)$

14. $\csc(\arctan 2x)$

15–16 ||| Graph the given functions on the same screen. How are these graphs related?

15. $y = \sin x, \ -\pi/2 \leqslant x \leqslant \pi/2; \quad y = \sin^{-1}x; \quad y = x$

16. $y = \tan x, \ -\pi/2 < x < \pi/2; \quad y = \tan^{-1}x; \quad y = x$

17. Prove Formula 6 for the derivative of $\cos^{-1}$ by the same method as for Formula 3.

18. (a) Prove that $\sin^{-1}x + \cos^{-1}x = \pi/2$.
(b) Use part (a) to prove Formula 6.

19. Prove that $\dfrac{d}{dx}(\cot^{-1}x) = -\dfrac{1}{1+x^2}$.

20. Prove that $\dfrac{d}{dx}(\sec^{-1}x) = \dfrac{1}{x\sqrt{x^2-1}}$.

21. Prove that $\dfrac{d}{dx}(\csc^{-1}x) = -\dfrac{1}{x\sqrt{x^2-1}}$.

22–35 ||| Find the derivative of the function. Simplify where possible.

22. $y = \sqrt{\tan^{-1}x}$

23. $y = \tan^{-1}\sqrt{x}$

24. $h(x) = \sqrt{1-x^2}\arcsin x$

25. $y = \sin^{-1}(2x+1)$

26. $f(x) = x\ln(\arctan x)$

27. $H(x) = (1+x^2)\arctan x$

28. $h(t) = e^{\sec^{-1}t}$

29. $y = \cos^{-1}(e^{2x})$

30. $y = x\cos^{-1}x - \sqrt{1-x^2}$

31. $y = \arctan(\cos\theta)$

32. $y = \tan^{-1}(x - \sqrt{1+x^2})$

33. $h(t) = \cot^{-1}(t) + \cot^{-1}(1/t)$

34. $y = \tan^{-1}\left(\dfrac{x}{a}\right) + \ln\sqrt{\dfrac{x-a}{x+a}}$

35. $y = \arccos\left(\dfrac{b + a\cos x}{a + b\cos x}\right), \quad 0 \leqslant x \leqslant \pi, \ a > b > 0$

36–37 ||| Find the derivative of the function. Find the domains of the function and its derivative.

36. $f(x) = \arcsin(e^x)$

37. $g(x) = \cos^{-1}(3-2x)$

38. Find y' if $\tan^{-1}(xy) = 1 + x^2y$.

39. If $g(x) = x\sin^{-1}(x/4) + \sqrt{16-x^2}$, find $g'(2)$.

40. Find an equation of the tangent line to the curve $y = 3\arccos(x/2)$ at the point $(1, \pi)$.

41–42 ||| Find $f'(x)$. Check that your answer is reasonable by comparing the graphs of f and f'.

41. $f(x) = e^{-x}\arctan x$

42. $f(x) = x\arcsin(1-x^2)$

43–46 ||| Find the limit.

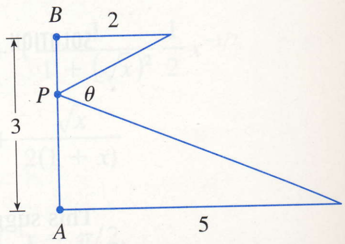
43. $\lim\limits_{x\to -1^+} \sin^{-1}x$

44. $\lim\limits_{x\to\infty} \arccos\left(\dfrac{1+x^2}{1+2x^2}\right)$

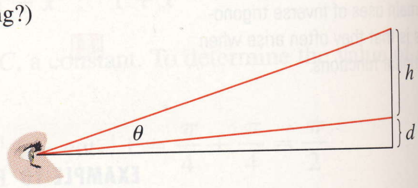
45. $\lim\limits_{x\to\infty} \arctan(e^x)$

46. $\lim\limits_{x\to 0^+} \tan^{-1}(\ln x)$

47. Where should the point P be chosen on the line segment AB so as to maximize the angle θ?



48. A painting in an art gallery has height h and is hung so that its lower edge is a distance d above the eye of an observer (as in the figure). How far from the wall should the observer stand to get the best view? (In other words, where should the observer stand so as to maximize the angle θ subtended at his eye by the painting?)



49. A ladder 10 ft long leans against a vertical wall. If the bottom of the ladder slides away from the base of the wall at a speed of 2 ft/s, how fast is the angle between the ladder and the wall changing when the bottom of the ladder is 6 ft from the base of the wall?

50. A lighthouse is located on a small island, 3 km away from the nearest point P on a straight shoreline, and its light makes four revolutions per minute. How fast is the beam of light moving along the shoreline when it is 1 km from P?

51–54 ||| Sketch the curve using the guidelines of Section 4.5.

51. $y = \sin^{-1}\left(\dfrac{x}{x+1}\right)$

52. $y = \tan^{-1}\left(\dfrac{x-1}{x+1}\right)$

53. $y = x - \tan^{-1}x$

54. $y = \tan^{-1}(\ln x)$

CAS **55.** If $f(x) = \arctan(\cos(3\arcsin x))$, use the graphs of f, f', and f'' to estimate the x-coordinates of the maximum and minimum points and inflection points of f.

56. Investigate the family of curves given by $f(x) = x - c\sin^{-1}x$. What happens to the number of maxima and minima as c changes? Graph several members of the family to illustrate what you discover.