If a is infinite, we let $t = 1/x$. Then $t \to 0^+$ as $x \to \infty$, so we have

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{t\to 0^+} \frac{f(1/t)}{g(1/t)}$$

$$= \lim_{t\to 0^+} \frac{f'(1/t)(-1/t^2)}{g'(1/t)(-1/t^2)} \qquad \text{(by l'Hospital's Rule for finite } a\text{)}$$

$$= \lim_{t\to 0^+} \frac{f'(1/t)}{g'(1/t)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}$$

7.7 Exercises

1–4 Given that

$$\lim_{x\to a} f(x) = 0 \qquad \lim_{x\to a} g(x) = 0 \qquad \lim_{x\to a} h(x) = 1$$

$$\lim_{x\to a} p(x) = \infty \qquad \lim_{x\to a} q(x) = \infty$$

which of the following limits are indeterminate forms? For those that are not an indeterminate form, evaluate the limit where possible.

1. (a) $\lim_{x\to a} \dfrac{f(x)}{g(x)}$ (b) $\lim_{x\to a} \dfrac{f(x)}{p(x)}$

(c) $\lim_{x\to a} \dfrac{h(x)}{p(x)}$ (d) $\lim_{x\to a} \dfrac{p(x)}{f(x)}$

(e) $\lim_{x\to a} \dfrac{p(x)}{q(x)}$

2. (a) $\lim_{x\to a} [f(x)p(x)]$ (b) $\lim_{x\to a} [h(x)p(x)]$

(c) $\lim_{x\to a} [p(x)q(x)]$

3. (a) $\lim_{x\to a} [f(x) - p(x)]$ (b) $\lim_{x\to a} [p(x) - q(x)]$

(c) $\lim_{x\to a} [p(x) + q(x)]$

4. (a) $\lim_{x\to a} [f(x)]^{g(x)}$ (b) $\lim_{x\to a} [f(x)]^{p(x)}$ (c) $\lim_{x\to a} [h(x)]^{p(x)}$

(d) $\lim_{x\to a} [p(x)]^{f(x)}$ (e) $\lim_{x\to a} [p(x)]^{q(x)}$ (f) $\lim_{x\to a} \sqrt[q(x)]{p(x)}$

5–62 Find the limit. Use l'Hospital's Rule where appropriate. If there is a more elementary method, consider using it. If l'Hospital's Rule doesn't apply, explain why.

5. $\lim_{x\to -1} \dfrac{x^2 - 1}{x + 1}$

6. $\lim_{x\to -2} \dfrac{x + 2}{x^2 + 3x + 2}$

7. $\lim_{x\to 1} \dfrac{x^9 - 1}{x^5 - 1}$

8. $\lim_{x\to 1} \dfrac{x^a - 1}{x^b - 1}$

9. $\lim_{x\to(\pi/2)^+} \dfrac{\cos x}{1 - \sin x}$

10. $\lim_{x\to 0} \dfrac{x + \tan x}{\sin x}$

11. $\lim_{t\to 0} \dfrac{e^t - 1}{t^3}$

12. $\lim_{t\to 0} \dfrac{e^{3t} - 1}{t}$

13. $\lim_{x\to 0} \dfrac{\tan px}{\tan qx}$

14. $\lim_{\theta\to \pi/2} \dfrac{1 - \sin\theta}{\csc\theta}$

15. $\lim_{x\to\infty} \dfrac{\ln x}{x}$

16. $\lim_{x\to\infty} \dfrac{e^x}{x}$

17. $\lim_{x\to 0^+} \dfrac{\ln x}{x}$

18. $\lim_{x\to\infty} \dfrac{\ln \ln x}{x}$

19. $\lim_{t\to 0} \dfrac{5^t - 3^t}{t}$

20. $\lim_{x\to 1} \dfrac{\ln x}{\sin \pi x}$

21. $\lim_{x\to 0} \dfrac{e^x - 1 - x}{x^2}$

22. $\lim_{x\to 0} \dfrac{e^x - 1 - x - (x^2/2)}{x^3}$

23. $\lim_{x\to\infty} \dfrac{e^x}{x^3}$

24. $\lim_{x\to 0} \dfrac{\sin x}{\sinh x}$

25. $\lim_{x\to 0} \dfrac{\sin^{-1}x}{x}$

26. $\lim_{x\to 0} \dfrac{\sin x - x}{x^3}$

27. $\lim_{x\to 0} \dfrac{1 - \cos x}{x^2}$

28. $\lim_{x\to\infty} \dfrac{(\ln x)^2}{x}$

29. $\lim_{x\to 0} \dfrac{x + \sin x}{x + \cos x}$

30. $\lim_{x\to 0} \dfrac{\cos mx - \cos nx}{x^2}$

31. $\lim_{x\to\infty} \dfrac{x}{\ln(1 + 2e^x)}$

32. $\lim_{x\to 0} \dfrac{x}{\tan^{-1}(4x)}$

33. $\lim_{x\to 1} \dfrac{1 - x + \ln x}{1 + \cos \pi x}$

34. $\lim_{x\to\infty} \dfrac{\sqrt{x^2 + 2}}{\sqrt{2x^2 + 1}}$

35. $\lim_{x\to 1} \dfrac{x^a - ax + a - 1}{(x - 1)^2}$

36. $\lim_{x\to 0} \dfrac{1 - e^{-2x}}{\sec x}$

37. $\lim_{x\to 0^+} \sqrt{x}\ln x$

38. $\lim_{x\to -\infty} x^2 e^x$

39. $\lim_{x\to 0} \cot 2x \sin 6x$

40. $\lim_{x\to 0^+} \sin x \ln x$

41. $\lim_{x\to\infty} x^3 e^{-x^2}$

42. $\lim_{x\to \pi/4} (1 - \tan x)\sec x$

43. $\lim_{x\to 1^+} \ln x \tan(\pi x/2)$

44. $\lim_{x\to\infty} x \tan(1/x)$

45. $\lim_{x\to 0} \left(\dfrac{1}{x} - \csc x\right)$

46. $\lim_{x\to 0} (\csc x - \cot x)$

47. $\lim_{x\to\infty} \left(\sqrt{x^2 + x} - x\right)$

48. $\lim_{x\to 1} \left(\dfrac{1}{\ln x} - \dfrac{1}{x-1}\right)$

49. $\lim_{x\to\infty} (x - \ln x)$

50. $\lim_{x\to\infty} (xe^{1/x} - x)$

51. $\lim_{x\to 0^+} x^{x^2}$

52. $\lim_{x\to 0^+} (\tan 2x)^x$

53. $\lim_{x\to 0} (1 - 2x)^{1/x}$

54. $\lim_{x\to\infty} \left(1 + \dfrac{a}{x}\right)^{bx}$

55. $\lim_{x\to\infty} \left(1 + \dfrac{3}{x} + \dfrac{5}{x^2}\right)^x$

56. $\lim_{x\to\infty} x^{(\ln 2)/(1+\ln x)}$

57. $\lim_{x\to\infty} x^{1/x}$

58. $\lim_{x\to\infty} (e^x + x)^{1/x}$

59. $\lim_{x\to\infty} \left(\dfrac{x}{x+1}\right)^x$

60. $\lim_{x\to 0} (\cos 3x)^{5/x}$

61. $\lim_{x\to 0^+} (\cos x)^{1/x^2}$

62. $\lim_{x\to\infty} \left(\dfrac{2x-3}{2x+5}\right)^{2x+1}$

63–64 ||| Use a graph to estimate the value of the limit. Then use l'Hospital's Rule to find the exact value.

63. $\lim_{x\to\infty} x[\ln(x+5) - \ln x]$

64. $\lim_{x\to \pi/4} (\tan x)^{\tan 2x}$

65–66 ||| Illustrate l'Hospital's Rule by graphing both $f(x)/g(x)$ and $f'(x)/g'(x)$ near $x = 0$ to see that these ratios have the same limit as $x \to 0$. Also calculate the exact value of the limit.

65. $f(x) = e^x - 1, \quad g(x) = x^3 + 4x$

66. $f(x) = 2x \sin x, \quad g(x) = \sec x - 1$

67–72 Use l'Hospital's Rule to help sketch the curve. Use the guidelines of Section 4.5.

67. $y = xe^{-x}$

68. $y = x(\ln x)^2$

69. $y = xe^{-x^2}$

70. $y = e^x/x$

71. $y = x - \ln(1 + x)$

72. $y = e^x - 3e^{-x} - 4x$

CAS **73–75** |||

(a) Graph the function.

(b) Use l'Hospital's Rule to explain the behavior as $x \to 0^+$ or as $x \to \infty$.

(c) Estimate the maximum and minimum values and then use calculus to find the exact values.

(d) Use a graph of f'' to estimate the x-coordinates of the inflection points.

73. $f(x) = x^{-x}$

74. $f(x) = (\sin x)^{\sin x}$

75. $f(x) = x^{1/x}$

76. Investigate the family of curves given by $f(x) = x^n e^{-x}$, where n is a positive integer. What features do these curves have in common? How do they differ from one another? In particular, what happens to the maximum and minimum points and inflection points as n increases? Illustrate by graphing several members of the family.

77. Investigate the family of curves given by $f(x) = xe^{-cx}$, where c is a real number. Start by computing the limits as $x \to \pm\infty$. Identify any transitional values of c where the basic shape changes. What happens to the maximum or minimum points and inflection points as c changes? Illustrate by graphing several members of the family.

78. The first appearance in print of l'Hospital's Rule was in the book *Analyse des Infiniment Petits* published by the Marquis de l'Hospital in 1696. This was the first calculus *textbook* ever published and the example that the Marquis used in that book to illustrate his rule was to find the limit of the function

$$y = \frac{\sqrt{2a^3x - x^4} - a\sqrt[3]{aax}}{a - \sqrt[4]{ax^3}}$$

as x approaches a, where $a > 0$. (At that time it was common to write aa instead of a^2.) Solve this problem.

79. If an initial amount A_0 of money is invested at an interest rate i compounded n times a year, the value of the investment after t years is

$$A = A_0\left(1 + \frac{i}{n}\right)^{nt}$$

If we let $n \to \infty$, we refer to the *continuous compounding* of interest. Use l'Hospital's Rule to show that if interest is compounded continuously, then the amount after n years is

$$A = A_0 e^{it}$$

80. If an object with mass m is dropped from rest, one model for its speed v after t seconds, taking air resistance into account,

$$v = \frac{mg}{c}\left(1 - e^{-ct/m}\right)$$

where g is the acceleration due to gravity and c is a positive constant. (In Chapter 10 we will be able to deduce this