Solutions of Assignment # 9.

Problem 1. Is the following definition equivalent to the definition of a Cauchy (fundamental) sequence? As usual, prove if YES, provide a counterexample if NO.

$$\forall \varepsilon > 0 \ \exists N \in \mathbb{N} \ \forall n \geq N \quad |x_n - x_{n+1}| < \varepsilon.$$

Solution.
Analysis Before to write the actual solution let us analyse the new definition and compare it with the old one. In the old (correct) definition we are asked to compare distances $|x_m - x_n|$ for all large m and n without any restrictions on relations between n and m. Here we compare only distances between n and $n+1$. In other words, in the new definition we are forced to take $m = n+1$. Of course, such an addition should spoil the property "to be fundamental": the new definition is much less restrictive. Therefore we will be trying to prove that the answer is NO and that there exists a sequence which satisfies the new definition (which is less restrictive), but fails to satisfy the old definition. In order to do that we notice that for all $m > n$ we have

$$x_m - x_n = (x_m - x_{m-1}) + (x_{m-1} - x_{m-2}) + \dots + (x_{n+2} - x_{n+1}) + (x_{m+1} - x_n).$$

In other words, we want to find a sequence such that all differences above are positive and small (to satisfy a new definition) but the sum is big for big enough m (of course, m should be much larger that n). So we have to find a sequence $y_n = x_{n+1} - x_n$ with two properties: sum of y_n's is big (actually unbounded), but each y_n is very small (actually goes to 0). We had an example of such a sequence in class: $1/n$ (remember, Harmonic series is divergent).
Actual Solution. We show that answer is NO by considering the following example. Let $\{x_n\}_{n=1}^{\infty}$ be defined by

$$x_n = \sum_{k=1}^{n} \frac{1}{k}.$$

In class we proved that $\{x_n\}_{n=1}^{\infty}$ is not fundamental. On the other hand, given $\varepsilon > 0$ we choose $N > 1/\varepsilon$. Then for all $n \geq N$ one has

$$|x_{n+1} - x_n| = \frac{1}{n+1} < \frac{1}{N} < \varepsilon,$$

which shows that the sequence satisfies the new definition. □

Answer. NO.

Problem 2. Let $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ be two convergent (in $\mathbb{R}$) sequences such that $x_n \geq y_n$ for all $n \in \mathbb{N}$. Show that

$$\lim_{n\to\infty} x_n \geq \lim_{n\to\infty} y_n.$$

Solution.
Analysis Again we start with analysis. We are going to argue by contradiction. From the definition of the limit we know that the tail of a convergent sequence is very close to the limit, namely it is in a small interval around the limit and it is up to us how small interval to choose. So if the first limit is strictly smaller that the second one, we will choose intervals around both limits in such a way that they don't intersect (draw a picture!). Then, clearly, we can't satisfy the condition.

Actual Proof. We argue by contradiction. Denote

$$a = \lim_{n\to\infty} x_n, \qquad b = \lim_{n\to\infty} y_n$$

and assume $a < b$. Choose $\varepsilon = (b-a)/3$ and apply the definition of the limit to both sequences. There exists N_1 such that for every $n \geq N_1$

$$|x_n - a| < \varepsilon,$$

and there exists N_2 such that for every $n \geq N_2$

$$|y_n - b| < \varepsilon.$$

Note

$$|x_n - a| < \varepsilon \quad \Leftrightarrow \quad x_n \in (a-\varepsilon, a+\varepsilon) \quad \Rightarrow \quad x_n \leq a+\varepsilon$$

and

$$|y_n - b| < \varepsilon \quad \Leftrightarrow \quad y_n \in (b-\varepsilon, b+\varepsilon) \quad \Rightarrow \quad y_n \geq b-\varepsilon.$$

Now take $N = \max\{N_1, N_2\}$. Then we have

$$x_N \leq a+\varepsilon \qquad \text{and} \qquad y_N \geq b-\varepsilon.$$

Finally note that $a+\varepsilon < b-\varepsilon$ (since $\varepsilon = (b-a)/3$, so $b-a > 2\varepsilon$), which implies $x_N < y_N$. It contradicts the condition on sequences. □

Problem 3. Using only the definition, prove that

$$\lim_{n\to\infty} (-n^2 + 10n + 100) = -\infty.$$

Solution. Fix $M \geq 0$. Choose $N > M + 100$. Then for every $n \geq N$ we have

$$-n^2 + 10n + 100 = -n(n-10) + 100 < -n + 100 \leq -N + 100 < -M$$

(in the first inequality we used $n \geq N > 100 > 11$). It proves the result. □

Problem 4. Let $x_n = \sqrt{n^2+4n+5} - n$. Is $\{x_n\}_{n=1}^{\infty}$ convergent? If YES find the limit, if NOT explain why.

Solution. We have

$$x_n = \frac{(\sqrt{n^2+4n+5}-n)(\sqrt{n^2+4n+5}+n)}{\sqrt{n^2+4n+5}+n} = \frac{n^2+4n+5-n^2}{\sqrt{n^2+4n+5}+n}$$

$$= \frac{4n+5}{\sqrt{n^2+4n+5}+n} = \frac{4+5/n}{\sqrt{1+4/n+5/n^2}+1}.$$

Since

$$\lim_{n\to\infty} 5/n^2 = \lim_{n\to\infty} 4/n = \lim_{n\to\infty} 5/n = 0,$$

as was discussed many times, and since

$$\lim_{n\to\infty} \sqrt{z_n} = \sqrt{\lim_{n\to\infty} z_n}$$

for a non-negative sequence, as was proved in one of previous assignments, we obtain that the limit of our sequence exists and

$$\lim_{n\to\infty} x_n = \frac{4+0}{\sqrt{1+0+0}+1} = 2.$$

□

Answer.

$$\lim_{n\to\infty} (\sqrt{n^2+4n+5} - n) = 2.$$

Problem 5. Let $x_n \geq 0$ for all $n \in \mathbb{N}$. Assume that

$$\sum_{n=1}^{\infty} x_n < \infty.$$

Prove that

$$\lim_{n\to\infty} x_n = 0.$$

Solution. Denote

$$S_m = \sum_{n=1}^{m} x_n.$$

Clearly $\{S_m\}_{m=1}^{\infty}$ is increasing, so it has limit and we are given that this limit is in $\mathbb{R}$ (recall here $\sum_{n=1}^{\infty} x_n = \lim_{m\to\infty} S_m$).

Way 1. Denote this limit by ℓ and let $T_m = S_{m+1}$. Clearly,

$$\lim_{m\to\infty} T_m = \lim_{m\to\infty} S_m = \ell$$

(because $\{T_m\}_{m=1}^{\infty}$ is a tail of $\{S_m\}_{m=1}^{\infty}$). Since for every n one has $x_n = T_n - S_n$ we obtain by the corresponding theorem that limit of $\{x_n\}_{n=1}^{\infty}$ exists and

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} T_n - \lim_{n\to\infty} S_n = \ell - \ell = 0.$$

Way 2. Since $\{S_m\}_{m=1}^{\infty}$ is convergent, by the Cauchy criterion, we obtain that $\{S_m\}_{m=1}^{\infty}$ is fundamental. Thus for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for every $n, m \geq N$ one has

$$|S_n - S_m| < \varepsilon.$$

In particular, taking $m = n+1$ we have for all $n \geq N$

$$|x_n| = |S_{n+1} - S_n| < \varepsilon.$$

It proves that for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for every $n \geq N$ one has $|x_n| = |S_{n+1} - S_n| < \varepsilon$, which means $x_n \to 0$ as $n \to \infty$. □