Solutions of Assignment # 8.

Problem 1. Is the following statement true or false?
a. If both $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are divergent sequences then $\{x_n + y_n\}_{n=1}^{\infty}$ is also divergent.
b. If both $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are divergent sequences then $\{x_n\ y_n\}_{n=1}^{\infty}$ is also divergent.
c. If $\{x_n\}_{n=1}^{\infty}$ is a convergent sequence and $\{y_n\}_{n=1}^{\infty}$ is a divergent sequence then $\{x_n + y_n\}_{n=1}^{\infty}$ is divergent.
d. If $\{x_n\}_{n=1}^{\infty}$ is a convergent sequence and $\{y_n\}_{n=1}^{\infty}$ is a divergent sequence then $\{x_n y_n\}_{n=1}^{\infty}$ is divergent.

Answer and solution.
a. False. Consider the sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ defined by $x_n = (-1)^n$ and $y_n = -x_n = (-1)^{n+1}$. From the class we know that $\{x_n\}_{n=1}^{\infty}$ is divergent and by the similar reason $\{y_n\}_{n=1}^{\infty}$ is divergent (note that $\{y_n\}_{n=1}^{\infty}$ is a tail of $\{x_n\}_{n=1}^{\infty}$). But $x_n + y_n = 0$ for every n. Thus $\{x_n + y_n\}_{n=1}^{\infty}$ is convergent to 0 (as a constant sequence).
b. False. Consider the sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ defined by $x_n = (-1)^n$ and $y_n = x_n = (-1)^n$. Then $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are divergent. But $x_n y_n = 1$ for every n. Thus $\{x_n y_n\}_{n=1}^{\infty}$ is convergent to 1 (as a constant sequence).
c. True. We prove it by contradiction. Assume that $\{x_n\}_{n=1}^{\infty}$ is a convergent sequence, $\{y_n\}_{n=1}^{\infty}$ is a divergent sequence but $\{x_n + y_n\}_{n=1}^{\infty}$ is convergent. Denote $z_n = x_n + y_n$ for every n. We have that $\{x_n\}_{n=1}^{\infty}$ and $\{z_n\}_{n=1}^{\infty}$ are convergent sequences. Then, by a theorem proved in class, we have that $\{z_n - x_n\}_{n=1}^{\infty}$ is a convergent sequence. It shows that $\{y_n\}_{n=1}^{\infty}$ is convergent, which contradicts our assumption.
d. False. Consider the sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ defined by $x_n = 0$ and $y_n = (-1)^n$. Then $\{x_n\}_{n=1}^{\infty}$ is convergent to 0 and $\{y_n\}_{n=1}^{\infty}$ is divergent. But $x_n y_n = 0$ for every n. Thus $\{x_n y_n\}_{n=1}^{\infty}$ is convergent to 0.

□

Problem 2. Let $\{x_n\}_{n=1}^{\infty}$ be defined as follows: $x_1 = 0$ and $x_{n+1} = \sqrt{2 + x_n}$ (for every $n \geq 1$). Prove that the sequence is convergent and find its limit.

Solution. We prove that $\{x_n\}_{n=1}^{\infty}$ is convergent to 2.

First, by induction, we show that $\{x_n\}_{n=1}^{\infty}$ is increasing and bounded. More precisely, we show that for every $n \geq 1$ one has $x_n < 2$ and $x_n < x_{n+1}$.
Base. The statement is true for $n = 1$, since $x_1 = 0 < 2$ and $x_2 = \sqrt{2} > x_1$.
Induction step. Assume the statement is correct for $n = k$, that is $x_k < 2$ and $x_k < x_{k+1}$. Then we have $x_{k+1} = \sqrt{2 + x_k} < \sqrt{2 + 2} = 2$; and

$$x_{k+1} = \sqrt{2 + x_k} < \sqrt{2 + x_{k+1}} = x_{k+2}.$$

Thus the statement is true for $n = k + 1$. So, by the induction, the statement is true for every n.

By a theorem proved in class, every increasing bounded above sequence is convergent. We obtain that $\{x_n\}_{n=1}^{\infty}$ is convergent. Denote its limit by ℓ. Clearly $\{x_{n+1}\}_{n=1}^{\infty}$ has the same limit (as a tail of a convergent sequence: in class we proved that any subsequence of a convergent sequence is convergent to the same limit). Now we use another theorem from the class, saying that the limit of the sum of sequences exists and equal to the sum of their limits, so

$$\lim_{n\to\infty} (2 + x_n) = 2 + \lim_{n\to\infty} x_n = 2 + \ell.$$

Finally, from the previous homework, we know that we can take square roots (we use also that $x_n \geq 0$, which in turn implies $\ell \geq 0$),

$$\ell = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \sqrt{2 + x_n} = \sqrt{\lim_{n\to\infty} (2 + x_n)} = \sqrt{2 + \ell}.$$

It implies $\ell^2 = 2 + \ell$. Solving this we obtain that either $\ell = 2$ or $\ell = -1$. But we know that $\ell \geq 0$. It implies that $\ell = 2$.

$\square$

Answer.

$$\lim_{n\to\infty} x_n = 2.$$

Problem 3. Let $\{x_n\}_{n=1}^{\infty}$ be defined by $x_n = \frac{\cos n}{n}$. Is $\{x_n\}_{n=1}^{\infty}$ convergent? If yes find the limit.

Solution. We show that

$$\lim_{n\to\infty} x_n = 0.$$

Indeed,

$$0 \leq |x_n| = \frac{|\cos n|}{n} \leq \frac{1}{n},$$

and

$$\lim_{n\to\infty} \frac{1}{n} = \lim_{n\to\infty} 0 = 0.$$

By the Squeeze Theorem

$$\lim_{n\to\infty} |x_n| = 0.$$

By the drill problem we obtain the desired result. $\square$

Answer.

$$\lim_{n\to\infty} x_n = 0.$$

Problem 4. Let $0 \leq a \leq b$. Let $\{x_n\}_{n=1}^{\infty}$ be defined by $x_n = (a^n + b^n)^{1/n}$. Prove that $\{x_n\}_{n=1}^{\infty}$ is convergent to b.

Solution. We clearly have $b^n \leq a^n + b^n \leq 2b^n$, which implies

$$b \leq (a^n + b^n)^{1/n} \leq 2^{1/n} b.$$

By a theorem proved in the class we have

$$\lim_{n\to\infty} 2^{1/n} = 1, \quad \text{which implies} \quad \lim_{n\to\infty} (2^{1/n} b) = b$$

Thus, by the Squeeze Theorem

$$\lim_{n\to\infty} (a^n + b^n)^{1/n} = b.$$

$\square$

Problem 5. Let $x_n = (1 + \frac{1}{n})^{n+1}$. Is $\{x_n\}_{n=1}^{\infty}$ convergent? If yes, find its limit.

Solution. First note

$$\lim_{n\to\infty} \frac{1}{n} = 0,$$

as was proved many times, so

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)=1.$$

Now recall that

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n=e,$$

and use a theorem from the class, saying that the limit of the product of sequences exists and equal to the product of their limits, so

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n+1}=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\lim_{n\to\infty}\left(1+\frac{1}{n}\right)=e.$$

$\square$

Answer.

$$\lim_{n\to\infty}x_n=e.$$